J. Math. Anal. Appl. 410 (2014) 151–157

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The Dirichlet divisor problem, traces and determinants for complex powers of the twisted bi-Laplacian

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ARTICLE INFO

Article history:
Received 22 January 2013
Available online 16 August 2013
Submitted by S. Fulling

Keywords:
Twisted bi-Laplacian
Dirichlet divisor problem
Counting function
Complex powers
Zeta function
Riemann zeta function
Trace
Heat semigroup
Dixmier trace
Inverse
Zeta function regularizations
Determinant

ABSTRACT

Estimating the counting function for the eigenvalues of the twisted bi-Laplacian leads to the Dirichlet divisor problem, which is then used to compute the trace of the heat semigroup and the Dixmier trace of the inverse of the twisted bi-Laplacian. The zeta function regularizations of the traces and determinants of complex powers of the twisted bi-Laplacian are computed. A formula for the zeta function regularizations of determinants of heat semigroups of complex powers of the twisted bi-Laplacian is given.

1. Introduction

The twisted Laplacian L on $\mathbb{R}^2$ is the second-order partial differential operator given by

$$L = -\Delta + \frac{1}{4}\left(x^2 + y^2\right) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Thus, the twisted Laplacian L is the Hermite operator

$$H = -\Delta + \frac{1}{4}\left(x^2 + y^2\right)$$

perturbed by the partial differential operator $-iN$, where

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.017

$$N = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the rotation operator.

That H is called the Hermite operator is due to the fact that Hermite functions are the eigenfunctions of H. See, for instance, Section 6.4 in [9]. That N is called the rotation operator can be attributed to the fact that in polar coordinates,

$$N = \frac{\partial}{\partial \theta},$$

which is the simplest differential operator on the unit circle centered at the origin.

The twisted Laplacian appears in harmonic analysis naturally in the context of Wigner transforms and Weyl transforms [2,12]. In the paper [1], it is shown that L is essentially self-adjoint, and the spectrum $\Sigma(L_0)$ of the closure L_0 is given by a sequence of eigenvalues, which are odd natural numbers, i.e.,

$$\Sigma(L_0) = \{2k+1:\ k = 0, 1, 2, \ldots\}.$$

It should be noted, however, that each eigenvalue has infinite multiplicity.

Renormalizing the twisted Laplacian L to the partial differential operator P given by

$$P = \frac{1}{2}(L+1), \tag{1.2}$$

we see that the eigenvalues of P are the natural numbers $1, 2, \ldots,$ and each eigenvalue, as in the case of L, has infinite multiplicity.

Now, the conjugate $\bar{L}$ of the twisted Laplacian L is given by

$$\bar{L} = -\Delta + \frac{1}{4}(x^2+y^2) + i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{1.3}$$

and after renormalization, we get the conjugate Q of P given by

$$Q = \frac{1}{2}(\bar{L}+1). \tag{1.4}$$

The aim of this paper is to analyze the heat kernels and Green functions of complex powers of the twisted bi-Laplacian M defined by

$$M = QP = PQ = \frac{1}{4}(H - iN + 1)(H + iN + 1), \tag{1.5}$$

where P and Q commute because it can be shown by easy computations that H and N commute, i.e., $HNf = NHf$ for all functions f in $C^\infty(\mathbb{R}^2)$.

It is proved in [3] that M is essentially self-adjoint on $L^2(\mathbb{R}^2)$. The unique self-adjoint extension of M on $L^2(\mathbb{R}^2)$ is again denoted by M.

In order to describe the spectral properties of M precisely, let us first recall that the Fourier–Wigner transform $V(f,g)$ of two functions f and g in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ is the function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$ given by

$$V(f,g)(q,p) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iqy} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}\,dy,$$

for all q and p in $\mathbb{R}$. For $k = 0, 1, 2, \ldots,$ the Hermite function e_k of order k is defined on $\mathbb{R}$ by

$$e_k(x) = \frac{1}{(2^k k!\sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2}\left(\frac{d}{dx}\right)^k e^{-x^2}, \quad x \in \mathbb{R}. \tag{1.7}$$

Now, for $j, k = 0, 1, 2, \ldots,$ we define the function $e_{j,k}$ on $\mathbb{R}^2$ by

$$e_{j,k}(x,y) = V(e_j, e_k)(x,y), \quad x, y \in \mathbb{R}. \tag{1.8}$$

It can be shown that $\{e_{j,k}:\ j, k = 0, 1, 2, \ldots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$. See, for example, Theorem 21.2 in [12].