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Existence and multiplicity of solutions for a class of elliptic boundary value problems

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ABSTRACT

In this paper, we investigate the existence and multiplicity of solutions for the following elliptic boundary value problems

$$\begin{cases} -\Delta u + a(x)u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g(x,u) = -K_u(x,u) + W_u(x,u)$. By using the symmetric mountain pass theorem, we obtain two results about infinitely many solutions when $g(x,u)$ is odd in u, K satisfies the pinching condition and W has a super-quadratic growth. Moreover, when the condition "$g(x,u)$ is odd" is not assumed, by using the mountain pass theorem, we also obtain two existence results of one nontrivial weak solution. One of these results generalizes a recent result in Mao, Zhu and Luan (2012) [10].

1. Introduction and main results

In this paper, we investigate the following elliptic boundary value problems

$$\begin{cases} -\Delta u + a(x)u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geqslant 3)$ with smooth boundary $\partial\Omega$, $g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and $a \in L^{N/2}(\Omega)$.

When $a(x) \equiv 0$, system (1.1) reduces to

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Via variational methods, there have been lots of contributions on existence and multiplicity of solutions for system (1.1) and (1.2), see [1–10] and references therein. It is well known that a famous super-quadratic condition is the Ambrosetti–Rabinowitz (AR) condition: there exist $\mu > 2$, $l_0 > 0$ such that

$$0 < \mu G(x,z) \leqslant z g(x,z), \quad \text{for all } |z| \geqslant l_0,\ x \in \Omega,$$

where $G(x,z) = \int_0^z g(x,s)\,ds$. The (AR)-condition has been extensively applied to study the existence and multiplicity of solutions for many differential systems, for example, Hamiltonian system and damped differential system, see [2,11–17].

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There are lots of super-quadratic functions which do not satisfy (AR)-condition, for example,

$$G(x,z) = |z|^2 \ln\bigl(1+|z|^2\bigr).$$

There have been some contributions which devoted to improve (AR)-condition, see [3,7–10,18–22]. For system (1.1), under more general conditions than (AR)-condition, recently, in [7], Jiang and Tang obtained that system (1.1) has a nontrivial solution by using a local linking theorem due to Li and Willem (see [6]) and in [8] and [9], those authors obtained system (1.1) has infinitely many solutions by using a variant of the Fountain theorem. Fountain theorem was obtained by Bartsch in [25]. Except that the case $G(x,z)$ is asymptotically-quadratic was also considered in [8], in all results in [7–9], $G(x,z)$ is super-quadratic. Recently, Mao, Zhu and Luan in [10] investigated system (1.2) under one new case that $G(x,z) = -K(x,z) + W(x,z)$, where K satisfies the pinching condition and W is super-quadratic, which is called mixed type nonlinearities. They obtained system (1.2) has a nontrivial weak solution.

In this paper, motivated by [3] and [10], we will investigate system (1.1) which is quite different from system (1.2). By using the symmetric mountain pass theorem, we obtain two results about infinitely many solutions when $g(x,u)$ is odd in u, K satisfies the pinching condition and W has a super-quadratic growth. Moreover, when the condition "$g(x,u)$ is odd" is not assumed, we also obtain two existence results of one nontrivial weak solution by using the mountain pass theorem. One of these results generalizes the result in [10] and our results are also different from those in [7–9] since we consider the mixed type nonlinearities. Next, we state our results.

Let

$$G(x,z) = \int_0^z g(x,s)\,ds, \qquad \tilde{W}(x,z) = \frac{1}{2} W_z(x,z)z - W(x,z).$$

Theorem 1.1. *Assume the following conditions hold:*

($\mathcal{L}_0$) $0 \notin \sigma(-\Delta + a)$, *where* $\sigma(-\Delta+a)$ *denotes the spectrum of* $-\Delta + a$;
(G1) $G(x,z) = -K(x,z) + W(x,z)$, $K, W : \Omega \times \mathbb{R}^1 \to \mathbb{R}^1$ *are* C^1*-maps;*
(G2) $g(x,u)$ *is odd in* u;
(K1) *there are two positive constants* b_1 *and* b_2 *such that*

$$b_1|z|^2 \leqslant K(x,z) \leqslant b_2|z|^2, \quad \text{for all } (x,z) \in \Omega \times \mathbb{R};$$

($\mathcal{B}$) $1 - 2b_2\tau_2^2 > 2b_1\tau_2^2$, *where* τ_2 *is the embedding constant in* $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$;
(K2) $0 \leqslant K_z(x,z)z \leqslant |K_z(x,z)||z| \leqslant 2K(x,z)$, *for all* $(x,z) \in \Omega \times \mathbb{R}$;
(W1) $\lim_{|z|\to 0} \frac{W_z(x,z)}{|z|} < 2b_1$ *uniformly in* x;
(W2) $W(x,0) \equiv 0$, $W(x,z) \geqslant 0$ *and* $W(x,z)/z^2 \to \infty$ *as* $|z| \to \infty$ *uniformly in* x;
(W3) $\tilde{W}(x,z) > 0$ *if* $z \neq 0$, $\tilde{W}(x,z) \to \infty$ *as* $|z| \to \infty$ *uniformly in x, and there exist* $r_0 > 0$ *and* $\sigma > N/2$ *such that* $|W_z(x,z)|^\sigma \leqslant c_0 \tilde{W}(x,z)|z|^\sigma$ *if* $|z| \geqslant r_0$.

Then system (1.1) has an unbounded sequence of solutions.

Note that in [21], Costa and Magalhães first used the condition $\tilde{W}(x,z) \to \infty$ as $|z| \to \infty$ uniformly in x, which is weaker than the (AR)-condition and useful in proving the Cerami condition (C) introduced by Cerami in [24].

If (W1) is strengthened to (W1)′ below, then ($\mathcal{B}$) in Theorem 1.1 can be relaxed to ($\mathcal{B}$)′ below.

Theorem 1.2. *Assume* ($\mathcal{L}_0$), (G1), (G2), (K1), (K2), (W2), (W3) *and the following conditions hold:*

(W1)′ $\lim_{|z|\to 0} \frac{W_z(x,z)}{|z|} = 0$ *uniformly in* x;
($\mathcal{B}$)′ $1 - 2b_2\tau_2^2 > 0$.

Then system (1.1) has an unbounded sequence of solutions.

If ($\mathcal{L}_0$) and the symmetric condition (G2) is deleted, we can show that system (1.1) has a nontrivial solution. To be precise, we have the following two theorems.

Theorem 1.3. *Assume that* (G1), (K1), (W1)′, (W2), (W3) *and the following conditions hold:*

(K2)′ $K(x,z) \leqslant K_z(x,z)z \leqslant |K_z(x,z)||z| \leqslant 2K(x,z)$, *for all* $(x,z) \in \Omega \times \mathbb{R}$;
($\mathcal{B}$)′′ $b_1 > 1$ *and there exists* θ *such that* $\max\{-\mu_1, 0\} < \theta < b_1 - 1$, *where* μ_1 *is the smallest eigenvalue of* $-\Delta + a$.

Then system (1.1) has a nontrivial solution.