J. Math. Anal. Appl. 410 (2014) 267–274

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence and uniqueness of solutions of reaction–convection equations with non-Lipschitz nonlinearity ☆

Jin Bo *, Su Ning

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China

ARTICLE INFO

Article history:
Received 20 March 2013
Available online 16 August 2013
Submitted by H.-M. Yin

Keywords:
Entropy
Subsolution
Supersolution
Maximal solution
Minimal solution
Comparison principle

ABSTRACT

In this work we study single balance law $u_t + \nabla \cdot \Phi(u) = f(u)$ with bounded initial value, and find that there may exist maximal and minimal solutions, if $f(u)$ is not Lipschitz continuous at $u = 0$. We also show that comparison principle is valid for such solutions, and the solutions may blow up or not under certain conditions. It is determined by the strength of source supply, as well as the competition between the source and flux.

1. Introduction

In this paper we study the following problem

$$\begin{cases} u_t + \nabla \cdot \Phi(u) = f(u) & \text{in } \mathbb{R}^n \times (0, T), \\ u = u_0 & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{IVP}$$

where functions $\Phi : \overline{\mathbb{R}}_+ \to \mathbb{R}^n$, $f : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are continuous, $u_0 \in L^\infty(\mathbb{R}^n)$ is nonnegative.

This problem can be used to explain the phenomenon of some chemical reaction evolved in a thin tube. The equation means that with the flowing of the material there also exists absorption or production. The latent application of this one possibly lies in the study of the fluid dynamics, the thermodynamics and the statistical mechanics. Furthermore, there's some equation sets which can be converted into a form of the conservation law studied in the publication on the reacting fluid flows (cf. [1]) and magnetohydrodynamics (cf. [8]). In some special cases, this differential equation set is a first-order quasilinear hyperbolic one. The hyperbolic structure admits the analysis on separate functions relating to the components of the unknown vector function after a suitable variable transformation. So the application is possible. Moreover, this problem also provides a model to characterize the asymptotic behavior for the solutions of the problems each having a viscous term in the equation.

It is well known (cf. [2,4–7]) that under mild conditions there exist unique (generalized, entropy) solutions to problem (IVP). Specifically, if $f(z)$ is uniformly Lipschitz continuous on $[0, \infty)$, then (IVP) has unique nonnegative, bounded solutions for any $T > 0$.

☆ This research is supported by the National Natural Science Foundation, PR China.
* Corresponding author.
E-mail address: jin-b09@mails.tsinghua.edu.cn (B. Jin).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.023

In this work we are interested in the case that $f(z)$ is not Lipschitz continuous at $z=0$ and/or $z=\infty$. Both non-uniqueness and finite-time blow-up may occur. In order to deal with the possible non-uniqueness, we utilize the notions of maximal and minimal solutions to problem (IVP), which were firstly introduced by de Pablo and Vazquez in [10,9] for reaction–diffusion equation

$$u_t = \Delta(u^m) + u^r \quad \text{in } \mathbb{R}^n \times (0,T),$$

in the case $m>0$, $r \in (0,1)$. It will be shown in the present article, under hypotheses which essentially consist of local Lipschitz continuity of $f(z)$ in $(0,\infty)$, that the Cauchy problem (IVP) has a maximal solution $\bar{u}(x,t)$ and a minimal solution $\underline{u}(x,t)$ on $\mathbb{R}^n \times (0,\tau)$ for some $\tau \in (0,T]$. Furthermore, if $\int_{0^+} dz/f(z) < \infty$, then non-uniqueness does occur. As to the blow-up issue, it is easy to verify that no blow-up occurs if $\int^\infty dz/f(z) = \infty$. It should be pointed, however, that $\int^\infty dz/f(z) < \infty$ does not always imply the blow-up. In fact, we will show that a strong convection may deflate the increase of the solutions so that the finite-time blow-up never appear.

This paper is organized as follows. In the next section we show the main results of this work, while the specific hypotheses and some definitions are given and explained. Then existence of the solutions to problem (IVP) is proved in Section 3. The method follows the idea in [10], but apparently with some vital modification. In Section 4, the results of uniqueness and comparison theorems are proved. At last, Section 5 is mainly focused on finite-time blow-up in some special cases.

2. Preliminaries and main results

Throughout this work, the following assumptions on $\Phi(z)$, $f(z)$ and $u_0(x)$ are imposed.

$$\begin{cases} \Phi(z) = (\varphi_1(z), \ldots, \varphi_n(z)), \quad \varphi_i(z) \in C(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+),\ i=1,\ldots,n; \\ \text{There exist concave, increasing functions } \sigma_i(z) \in C(\overline{\mathbb{R}}_+),\ \text{satisfying} \\ \sigma_i(0)=0, \quad |\varphi_i(z) - \varphi_i(z')| \leqslant \sigma_i(|z-z'|), \quad \text{for } z,z' \in \mathbb{R},\ i=1,\ldots,n; \\ \liminf_{z\to 0} z^{1-n} \prod_{i=1}^n \sigma_i(z) = 0, \end{cases} \tag{H-1}$$

$$f(z) \in C(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+), \quad \text{satisfying } f(0)=0, \tag{H-2}$$

$$u_0(x) \in L^\infty(\mathbb{R}^n), \quad \text{with } u_0 \geqslant 0 \text{ a.e. } \mathbb{R}^n. \tag{H-3}$$

It should be noted that condition (**H-1**) is necessary for uniqueness of solutions even in the case $f(z) \equiv 0$. A counter-example when $n=2$ and without this restriction is demonstrated in [7]. This restriction is also necessary for uniqueness of solutions of stationary conservation problems, see [2]. Condition (**H-2**) means that $f(z)$ is admitted not Lipschitz continuous near $z=0$ and $z=\infty$.

Now we recall the notion of solutions. Denote

$$z^+ := \max\{z,0\}, \quad z^- := (-z)^+.$$

Definition 2.1. (Cf. [2], also [6] and [7].) Let $Q = \mathbb{R}^n \times (0,T)$. A function $u(x,t) \in L^\infty(Q)$ is called a (generalized, entropy) subsolution (resp. supersolution) of problem (IVP), if $u \geqslant 0$ a.e. and for any $k \in \mathbb{R}$,

$$(u-k)_t^+ + \nabla \cdot [\operatorname{sgn}(u-k)^+(\Phi(u) - \Phi(k))] \leqslant \operatorname{sgn}(u-k)^+ f(u) \quad \text{in } D'(Q),$$
$$\left(\text{resp. } (u-k)_t^- + \nabla \cdot [\operatorname{sgn}(u-k)^-(\Phi(u) - \Phi(k))] \leqslant -\operatorname{sgn}(u-k)^- f(u) \quad \text{in } D'(Q)\right);$$

and for all compact sets $K \subset \mathbb{R}^n$,

$$\operatorname{ess}\lim_{t\to 0^+} \int_K (u(x,t) - u_0(x))^+ \, dx = 0,$$
$$\left(\text{resp. ess}\lim_{t\to 0^+} \int_K (u(x,t) - u_0(x))^- \, dx = 0\right).$$

$u(x,t)$ is called a solution of (IVP) if it is both a subsolution and a supersolution.

The main results of this paper are as follows:

Theorem 2.2 *(Existence). There exists $T>0$ so that the problem (IVP) admits a maximal solution $\bar{u}(x,t)$ and a minimal solution $\underline{u}(x,t)$ in $\mathbb{R}^n \times (0,T)$. It means that if $u(x,t)$ is a solution of (IVP) in $\mathbb{R}^n \times (0,T)$, then*

$$\underline{u}(x,t) \leqslant u(x,t) \leqslant \bar{u}(x,t) \quad \text{for almost all } (x,t) \in \mathbb{R}^n \times (0,T).$$

In addition, if $\int^\infty dz/f(z) = \infty$, then problem (IVP) admits solutions for any $T>0$.