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Revised CPA method to compute Lyapunov functions for nonlinear systems

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ABSTRACT

The CPA method uses linear programming to compute Continuous and Piecewise Affine Lyapunov functions for nonlinear systems with asymptotically stable equilibria. In [14] it was shown that the method always succeeds in computing a CPA Lyapunov function for such a system. The size of the domain of the computed CPA Lyapunov function is only limited by the equilibrium's basin of attraction. However, for some systems, an arbitrary small neighborhood of the equilibrium had to be excluded from the domain a priori. This is necessary, if the equilibrium is not exponentially stable, because the existence of a CPA Lyapunov function in a neighborhood of the equilibrium is equivalent to its exponential stability as shown in [11]. However, if the equilibrium is exponentially stable, then this was an artifact of the method. In this paper we overcome this artifact by developing a revised CPA method. We show that this revised method is always able to compute a CPA Lyapunov function for a system with an exponentially stable equilibrium. The only conditions on the system are that it is C^2 and autonomous. The domain of the CPA Lyapunov function can be any a priori given compact neighborhood of the equilibrium which is contained in its basin of attraction. Whereas in a previous paper [10] we have shown these results for planar systems, in this paper we cover general n-dimensional systems.

1. Introduction

Lyapunov functions, first introduced in [23], are a fundamental tool to determine the stability of equilibria and their basins of attraction. They can be used for very general systems, e.g. nonautonomous systems [22,35,16], arbitrary switched nonautonomous systems [15], or differential inclusions [5], but in this paper we concentrate on autonomous systems.

Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and assume that the origin is an exponentially stable equilibrium of the system. Denote by $\mathcal{A}$ its basin of attraction. The standard method to verify the exponential stability of the origin is to solve the Lyapunov equation, i.e. to find a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ that is a solution to $J^T Q + Q J = -P$, where $J := D\mathbf{f}(\mathbf{0})$ is the Jacobian of $\mathbf{f}$ at the origin and $P \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite matrix. Then the function $\mathbf{x} \mapsto \mathbf{x}^T Q \mathbf{x}$ is a local Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, i.e. it is a Lyapunov function for the system in some neighborhood of the origin, cf. e.g. Theorem 4.7 in [22]. The size of this neighborhood is a priori not known and is, except for linear $\mathbf{f}$, in general a poor estimate of $\mathcal{A}$, cf. [13]. This method to compute local Lyapunov functions is constructive because there is an algorithm to solve the Lyapunov equation that succeeds whenever it possesses a solution, cf. Bartels and Stewart [3].

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The construction of Lyapunov functions for true nonlinear systems is a much harder problem than for linear systems. However, it has been studied intensively in the last decades and there have been numerous proposals of how to construct Lyapunov functions numerically. To name a few, Johansson and Rantzer proposed a construction method in [18] for piecewise quadratic Lyapunov functions for piecewise affine autonomous systems. In [7], Eghbal, Pariz, and Karimpour formulate the computation of piecewise quadratic Lyapunov functions for planar piecewise affine systems as linear matrix inequalities. In [32], Ratschan and She give an interval based branch-and-relax algorithm to compute polynomial Lyapunov-like functions for polynomial ODE. Another approach to numerically investigate the stability of nonlinear systems is, for example, given by Oishi in [27], where he considers the probabilistic computation of a stable control for systems that are parameter dependent, linear, and discrete. He uses a parameter dependent Lyapunov function.

Julian, Guivant, and Desages [20] and Julian [19] present a linear programming problem to construct piecewise affine Lyapunov functions for autonomous piecewise affine systems. This method can be used for autonomous, nonlinear systems if some a posteriori analysis of the generated Lyapunov function is done. In [17], Johansen uses linear programming to parameterize Lyapunov functions for autonomous nonlinear systems, but does not give error estimates. In [33], Rezaiee-Pajand and Moghaddasie proposed a different collocation method using two classes of basis functions. Giesl [8] proposed a method to construct Lyapunov functions for autonomous systems with an exponentially stable equilibrium by numerically solving a generalized Zubov equation, cf. [36]. A solution to Zubov's equation is a Lyapunov function for the system. He uses radial basis functions to approximate the solution and derives error estimates.

Parrilo [29] and Papachristodoulou and Prajna [28] consider the numerical construction of Lyapunov functions that can be expressed as sum of squares (SOS) of polynomials for autonomous polynomial systems. These ideas have been taken further by recent publications of Peet [30] and Peet and Papachristodoulou [31], where the existence of a polynomial Lyapunov function on bounded regions for exponentially stable systems is established. The Lyapunov functions are computed by means of convex optimization and are true Lyapunov functions and not approximations.

A complete Lyapunov function, first introduced by Conley in [6], is a generalization of a Lyapunov function for compact invariant sets, as discussed here, to an object completely characterizing the decomposition of a flow into a chain-recurrent and a gradient-like part. Norton [26] even suggested that this characterization should be referred to as the Fundamental Theorem of Dynamical Systems. In [21], Kalies, Mischaikow and VanderVorst present an algorithmic approach to construct approximations to complete Lyapunov functions for discrete dynamical systems. By considering the time-T map of a continuous system, this method can be used to find an approximation to a complete Lyapunov function for a continuous dynamical system as well. In [2], Ban and Kalies implement this algorithm and give examples of computed Lyapunov functions.

In [25], Hafstein (alias Marinosson) presents a method to compute piecewise affine Lyapunov functions. In this method one first triangulates a compact neighborhood $\mathcal{C} \subset \mathcal{A}$ of the origin and then constructs a linear programming problem with the property that a continuous Lyapunov function V, affine on each n-simplex of the triangulation, i.e. a CPA Lyapunov function, can be constructed from any feasible solution to it. In [13] it was proved that for exponentially stable equilibria this method is always capable of generating a Lyapunov function $V : \mathcal{C} \setminus \mathcal{N} \longrightarrow \mathbb{R}$, where $\mathcal{N} \subset \mathcal{C}$ is an arbitrary small, a priori determined neighborhood of the origin. In [14], these results were generalized to asymptotically stable systems, in [15] to asymptotically stable, arbitrary switched, nonautonomous systems, and in [1] to asymptotically stable differential inclusions.

In [9], the authors showed that the triangulation scheme used in [25,13–15] does in general generate suboptimal triangles at the equilibrium. However, in the same paper they proposed a new, fan-like triangulation around the equilibrium, and proved that a piecewise linear Lyapunov function with respect to this new triangulation always exists for planar systems. In [10], the authors showed how to compute a CPA Lyapunov function algorithmically for planar systems by using linear optimization. The modification to the algorithm in [15] is to use a fine, fan-like triangulation around the equilibrium, as suggested in [9]. The general n-dimensional case was treated in [11], where the authors proved, using different methods than in [9], that a piecewise linear Lyapunov function with respect to a modified, fan-like triangulation around the equilibrium always exists. However, the proof was non-constructive and it was not shown how to explicitly compute such a function. In this paper, the authors finish the work from [9–11] and deliver an algorithm to **compute** a CPA Lyapunov functions in n-dimensions and prove that the algorithm always succeeds in a finite number of steps whenever the system possesses an exponentially stable equilibrium.

The numerical discretization method presented in this paper is somewhat unusual since it is exact, i.e. it computes a true Lyapunov function and not an approximation. This is possible since a Lyapunov function is characterized through inequalities rather than equalities. Some other methods to construct Lyapunov functions, for example, the SOS method in [30,31], also share this property. It should, however, be noted that the interplay between continuous systems and their discretization is very well understood. In particular, many important dynamical properties like attractors and basins of attraction are inherited by discretization, even for control systems. For a detailed discussion of this see the important work of Grüne [12].

Let us give an overview over the contents: In Section 2 we define a linear programming problem in Definition 6 and show that a solution of this problem parameterizes a CPA Lyapunov function in Theorem 1. In Section 3, we explain in Definition 17 how to algorithmically find a suitable triangulation for the linear programming problem from Definition 6. The main result is Theorem 5, showing that the algorithm from Definition 17 always succeeds in computing a CPA Lyapunov function for a system with an exponentially stable equilibrium. In Section 5 we give examples of CPA Lyapunov functions computed by our method. The paper ends with some concluding remarks in Section 6.