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Connections between discriminants and the root distribution of polynomials with rational generating function

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ABSTRACT

Let $H_m(z)$ be a sequence of polynomials whose generating function $\sum_{m=0}^{\infty} H_m(z)t^m$ is the reciprocal of a bivariate polynomial $D(t,z)$. We show that in the three cases $D(t,z) = 1 + B(z)t + A(z)t^2$, $D(t,z) = 1 + B(z)t + A(z)t^3$ and $D(t,z) = 1 + B(z)t + A(z)t^4$, where $A(z)$ and $B(z)$ are any polynomials in z with complex coefficients, the roots of $H_m(z)$ lie on a portion of a real algebraic curve whose equation is explicitly given. The proofs involve the q-analogue of the discriminant, a concept introduced by Mourad Ismail.

1. Introduction

In this paper we study the root distribution of a sequence of polynomials satisfying one of the following three-term recurrences:

$$H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-2}(z) = 0,$$
$$H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-3}(z) = 0,$$
$$H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-4}(z) = 0,$$

with certain initial conditions and $A(z), B(z)$ polynomials in z with complex coefficients. For the study of the root distribution of other sequences of polynomials that satisfy three-term recurrences, see [8] and [11]. In particular, we choose the initial conditions so that the generating function is

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{D(t,z)}$$

where $D(t,z) = 1 + B(z)t + A(z)t^2$, $D(t,z) = 1 + B(z)t + A(z)t^3$, or $D(t,z) = 1 + B(z)t + A(z)t^4$. We notice that the root distribution of $H_m(z)$ will be the same if we replace 1 in the numerator by any monomial $N(t,z)$. If $N(t,z)$ is not a monomial, the root distribution will be different. The quadratic case $D(t,z) = 1 + B(z)t + A(z)t^2$ is not difficult and it is also mentioned in [13]. We present this case in Section 2 because it gives some directions to our main cases, the cubic and quartic denominators $D(t,z)$, in Sections 3 and 4.

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Our approach uses the concept of q-analogue of the discriminant (q-discriminant) introduced by Ismail [12]. The q-discriminant of a polynomial $P_n(x)$ of degree n and leading coefficient p is

$$\mathrm{Disc}_x(P;q) = p^{2n-2}q^{n(n-1)/2}\prod_{1\leqslant i<j\leqslant n}\left(q^{-1/2}x_i - q^{1/2}x_j\right)\left(q^{1/2}x_i - q^{-1/2}x_j\right) \tag{1}$$

where x_i, $1\leqslant i\leqslant n$, are the roots of $P_n(x)$. This q-discriminant is 0 if and only if a quotient of roots x_i/x_j equals q. As $q\to 1$, this q-discriminant becomes the ordinary discriminant which is denoted by $\mathrm{Disc}_x P(x)$. For the study of resultants and ordinary discriminants and their various formulas, see [1,2,9,10].

We will see that the concept of q-discriminant is useful in proving connections between the root distribution of a sequence of polynomials $H_m(z)$ and the discriminant of the denominator of its generating function $\mathrm{Disc}_t D(t,z)$. We will show in the three cases mentioned above that the roots of $H_m(z)$ lie on a portion of a real algebraic curve (see Theorem 1, Theorem 3, and Theorem 5). For the study of sequences of polynomials whose roots approach fixed curves, see [5–7]. Other studies of the limits of zeros of polynomials satisfying a linear homogeneous recursion whose coefficients are polynomials in z are given in [3,4]. The q-discriminant will appear as the quotient q of roots in t of $D(t,z)$. One advantage of looking at the quotients of roots is that, at least in the three cases above, although the roots of $H_m(z)$ lie on a curve depending on $A(z)$ and $B(z)$, the quotients of roots $t=t(z)$ of $D(t,z)$ lie on a fixed curve independent of these two polynomials. We will show that this independent curve is the unit circle in the quadratic case and two peculiar curves (see Figs. 1 and 2 in Sections 3 and 4) in the cubic and quartic cases. From computer experiments, this curve looks more complicated in the quintic case $D(z,t)=1+B(z)t+A(z)t^5$ (see Fig. 3 in Section 4).

As an application of these theorems, we will consider an example where $D(t,z)=1+(z^2-2z+a)t+z^2t^2$ and $a\in\mathbb{R}$. We will see that the roots of $H_m(z)$ lie either on portions of the circle of radius $\sqrt{a}$ or real intervals depending on the value a compared to the critical values 0 and 4. Also, the endpoints of the curves where the roots of $H_m(z)$ lie are roots of $\mathrm{Disc}_t D(t,z)$. Interestingly, the critical values 0 and 4 are roots of the double discriminant $\mathrm{Disc}_z\,\mathrm{Disc}_t D(t,z)=4096a^3(a-4)$.

2. The quadratic denominator

In this section, we will consider the root distribution of $H_m(z)$ when the denominator of the generating function is $D(t,z)=1+B(z)t+A(z)t^2$.

Theorem 1. *Let $H_m(z)$ be a sequence of polynomials whose generating function is*

$$\sum H_m(z)t^m = \frac{1}{1+B(z)t+A(z)t^2}$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z)\neq 0$ lie on the curve $\mathcal{C}_2$ defined by

$$\Im\frac{B^2(z)}{A(z)}=0 \quad\text{and}\quad 0\leqslant\Re\frac{B^2(z)}{A(z)}\leqslant 4,$$

and are dense there as $m\to\infty$.

Proof. Suppose z_0 is a root of $H_m(z)$ which satisfies $A(z_0)\neq 0$. Let $t_1=t_1(z_0)$ and $t_2=t_2(z_0)$ be the roots of $D(t,z_0)$. If $t_1=t_2$ then $\mathrm{Disc}_t D(t,z_0)=B^2(z_0)-4A(z_0)=0$. In this case z_0 belongs to $\mathcal{C}_2$, and we only need to consider the case $t_1\neq t_2$. By partial fractions, we have

$$\begin{aligned}\frac{1}{D(t,z_0)} &= \frac{1}{A(z_0)(t-t_1)(t-t_2)}\\ &= \frac{1}{A(z_0)(t_1-t_2)}\left(\frac{1}{t-t_1}-\frac{1}{t-t_2}\right)\\ &= \frac{1}{A(z_0)}\sum_{m=0}^{\infty}\frac{t_1^{m+1}-t_2^{m+1}}{(t_1-t_2)t_1^{m+1}t_2^{m+1}}t^n.\end{aligned} \tag{2}$$

Thus if we let $t_1=qt_2$ then q is an $(m+1)$-st root of unity and $q\neq 1$. By the definition of q-discriminant in (1), q is a root of $\mathrm{Disc}_t(D(t,z_0);q)$ which equals

$$q\left(B^2(z_0)-\left(q+q^{-1}+2\right)A(z_0)\right).$$

This implies that

$$\frac{B^2(z_0)}{A(z_0)}=q+q^{-1}+2.$$

Thus $z_0\in\mathcal{C}_2$ since q is an $(m+1)$-th root of unity.