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# On the group classification of systems of two linear second-order ordinary differential equations with constant coefficients

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ABSTRACT

The completeness of the group classification of systems of two linear second-order ordinary differential equations with constant coefficients is delineated in the paper. The new cases extend what has been done in the literature. These cases correspond to the type of equations where the commutative property of the coefficient matrices with respect to the dependent variables and the first-order derivatives in the considered system does not hold. A discussion of the results as well as a note on the extension to linear systems of second-order ordinary differential equations with more than two equations are given.



## 1. Introduction

Systems of second-order ordinary differential equations appear in the study of many applications. The study of their symmetry properties has generated sufficient interest in the literature. Here the group classification of systems of two linear second-order ordinary differential equations with constant coefficients is considered.

The literature has dealt extensively with symmetry properties of a scalar ordinary differential equation. Lie [10] himself classified all second-order ordinary differential equations with respect to complex Lie algebras [11]. Later on, in 1992, Gonzalez-Lopez et al. ordered the Lie classification of realizations of complex Lie algebras and extended it to the real case [6]. This extension allows one to classify a scalar ordinary differential equation with respect to the admitted Lie algebra. A large amount of results on the dimension and structure of symmetry algebras of linearizable ordinary differential equations is well known (see, for example, [16,7,12,8,9] and references therein).

The group classification of systems of second-order ordinary differential equations is less studied (see [5,18,17,14,3,4,2,1,15] and references therein).

In recent works [17,14,3,4,2] the authors focused on the study of systems of second-order ordinary differential equations with constant coefficients of the form

$$\mathbf{y}'' = M\mathbf{y}, \tag{1}$$

where $M$ is a matrix with constant entries. In the present paper it is shown that these types of systems do not exhaust a set of all systems of linear second-order ordinary differential equations with constant coefficients and the complete classification of linear systems of two second-order ordinary differential equations with constant coefficients is presented.

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As far as we are aware the results found here are new and have not been reported in the literature.

The paper is organized as follows. In the first part we present the simplification of a linear system of second-order equations with constant coefficients. This is then followed by a preliminary study of linear systems of second-order ordinary differential equations with constant coefficients. The later part is followed by a complete treatment of linear systems of second-order ordinary differential equations with constant coefficients as well as the discussion of the results and conclusion in the final part of the paper.

## 2. Simplification of a system of linear equations

Let a linear system of second-order equations with constant coefficients be given by:

$$\mathbf{y}'' = A\mathbf{y}' + B\mathbf{y} + \mathbf{f}, \tag{2}$$

where $A$ and $B$ are constant matrices. Using a particular solution $\mathbf{y}_p$, one can reduce (2) to the homogeneous system

$$\mathbf{y}'' = A\mathbf{y}' + B\mathbf{y}. \tag{3}$$

### 2.1. Linear change of the dependent variables

Applying the change,

$$\mathbf{y} = C\mathbf{y}_1,$$

where $C = C(x)$ is a nonsingular matrix, system (3) becomes

$$\mathbf{y}_1'' = \bar{A}\mathbf{y}_1' + \bar{B}\mathbf{y}_1, \tag{4}$$

where

$$\bar{A} = C^{-1}\left(AC - 2C'\right), \qquad \bar{B} = C^{-1}\left(BC + AC' - C''\right).$$

If one chooses the matrix $C(x)$ such that

$$C' = \frac{1}{2}AC,$$

then

$$\bar{B} = C^{-1}\left(BC + \frac{1}{2}A^2C - \frac{1}{4}A^2C\right) = C^{-1}\left(B + \frac{1}{4}A^2\right)C.$$

The existence of the nonsingular matrix $C(x)$ is guaranteed by the existence of the solution of the Cauchy problem

$$C' = \frac{1}{2}AC, \qquad C(0) = E,$$

where $E$ is the unit matrix. This solution defines the matrix

$$C(x) = e^{\frac{x}{2}A},$$

which commutes with the matrix $A$:

$$AC = CA.$$

Let us study the case when the matrix $\bar{B}$ is constant. First of all notice that

$$\frac{d}{dx}\left(C^{-1}\right) = -C^{-1}C'C^{-1}.$$

Then one has

$$\frac{d}{dx}\bar{B} = C^{-1}(BA - AB)C.$$

Hence, the matrix $\bar{B}$ is constant if and only if the matrices $A$ and $B$ commute

$$AB = BA. \tag{5}$$

The papers cited in the literature [17,14,3,4,2], where the group classification of systems of linear equations with constant coefficients was considered, studied systems with constant matrix $\bar{B}$. Thus, even for systems of linear equations with constant coefficients there is no complete study of the group classification. The present paper fills up this gap.