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A note on moment integrals and some applications

Jian Cao

Department of Mathematics, Hangzhou Normal University, Hangzhou City, Zhejiang Province, 310036, PR China

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ABSTRACT

In this paper, two moment integrals are given by the method of transformation. In addition, generalizations of Sears's transformation are obtained by moment integrals. Moreover, certain q-Mehler formulas for Rogers–Szegö polynomials are gained by moment integrals. Besides, an open problem of trilinear generating function is deduced by moment integrals. At last, generalizations of $U(n+1)$ type Kalnins–Miller transformation are achieved by moment integrals.

1. Introduction

In this paper, we follow the notations and terminology in [13] and suppose that $0<q<1$. The q-series and its compact factorials are defined respectively by

$$(a;q)_0=1,\qquad (a;q)_n=\prod_{k=0}^{n-1}\left(1-aq^k\right),\qquad (a;q)_\infty=\prod_{k=0}^{\infty}\left(1-aq^k\right) \tag{1.1}$$

and $(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$, where m is a positive integer and n is a nonnegative integer or ∞. In the context, convergence of basic hypergeometric series is no issue at all because they are the terminating q-series.

The basic hypergeometric series ${}_r\phi_s$ [13, Eq. (1.2.22)] is given by

$${}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}z^n\left[(-1)^nq^{n(n-1)/2}\right]^{s+1-r}, \tag{1.2}$$

which is convergent for either $|q|<1$ and $|z|<\infty$ when $r\leqslant s$ or $|q|<1$ and $|z|<1$ when $r=s+1$, provided that no zero appears in the denominator.

The Rogers–Szegö polynomials [25,28]

$$h_n(x|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}x^k\quad\text{and}\quad g_n(x|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}q^{k(k-n)}x^k=h_n\left(x|q^{-1}\right) \tag{1.3}$$

are closely related to the continuous q-Hermite polynomials via $H_n(\cos\theta|q)=e^{-in\theta}h_n(e^{2i\theta}|q)$, which play important roles in the theory of orthogonal polynomials.

E-mail addresses: 21caojian@gmail.com, 21caojian@163.com.

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Al-Salam and Carlitz [1] defined the following moment of two discrete distribution $\mathrm{d}\alpha^{(a)}(x)$ and $\mathrm{d}\beta^{(a)}(x)$ by Rogers–Szegö polynomials

$$\int_{-\infty}^{\infty} x^n\,\mathrm{d}\alpha^{(a)}(x) = h_n(a|q), \qquad \int_{-\infty}^{\infty} x^n\,\mathrm{d}\beta^{(a)}(x) = g_n(a|q), \tag{1.4}$$

where $\alpha^{(a)}(x)$ is a step function whose jumps occur at the points q^k and aq^k for $k \in \mathbb{N}$, while the jumps of $\beta^{(a)}(x)$ occur at the points q^{-k} for $k \in \mathbb{N}$. These jumps are given by, see [14] for the precise result,

$$\mathrm{d}\alpha^{(a)}\big(q^k\big) = \frac{q^k}{(a;q)_\infty (q,q/a;q)_k}, \qquad \mathrm{d}\alpha^{(a)}\big(aq^k\big) = \frac{q^k}{(1/a;q)_\infty (q,aq;q)_k}, \qquad \mathrm{d}\beta^{(a)}\big(q^{-k}\big) = \frac{a^k q^{k^2}(aq^{k+1};q)_\infty}{(q;q)_k}. \tag{1.5}$$

For more information about moment integral, please refer to [1,17].

Liu [21, Eq. (4.20)] utilized the technique of partial fraction to gain the following bivariate Rogers–Szegö polynomials $h_n(a,b|q)$

$$h_n(a,b|q) = \frac{a^n}{(b/a;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{(n+1)k}}{(q,aq/b;q)_k} + \frac{b^n}{(a/b;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{(n+1)k}}{(q,qb/a;q)_k}. \tag{1.6}$$

In this way, it's natural to define the generalized discrete probability measure $\alpha^{(a,b)}$ and $\beta^{(a,b)}$ by

$$\mathrm{d}\alpha^{(a,b)}\big(aq^k\big) = \frac{q^k}{(b/a;q)_\infty (q,aq/b;q)_k}, \qquad \mathrm{d}\alpha^{(a,b)}\big(bq^k\big) = \frac{q^k}{(a/b;q)_\infty (q,qb/a;q)_k},$$

$$\mathrm{d}\beta^{(a,b)}\big(q^{-k}\big) = \frac{a^k b^{-k} q^{k^2}(aq^{k+1}/b;q)_\infty}{(q;q)_k},$$

where

$$h_n(a,b|q) = \int_{-\infty}^{\infty} x^n\,\mathrm{d}\alpha^{(a,b)}(x) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix} a^{n-k}b^k, \qquad g_n(a,b|q) = \int_{-\infty}^{\infty} x^n\,\mathrm{d}\beta^{(a,b)}(x) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix} q^{k(k-n)}a^{n-k}b^k \tag{1.7}$$

and [5, Eq. (3.6) and p. 54]

$$\int_{-\infty}^{\infty}\frac{1}{(ax,bx;q)_\infty}\,\mathrm{d}\alpha^{(s,t)}(x) = \frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}, \qquad \int_{-\infty}^{\infty}(ax,bx;q)_\infty\,\mathrm{d}\beta^{(s,t)}(x) = \frac{(as,at,bs,bt;q)_\infty}{(abst/q;q)_\infty}. \tag{1.8}$$

The method of moment integral shows that it is an effective way to solve related problems. For more information, please refer to [2,1,14,15,18].

In this paper, we give the following two moment integrals.

Theorem 1. *For $m,n \in \mathbb{N}$ and $\max\{|as|,|at|,|bs|,|bt|\} < 1$, we have*

$$\int_{-\infty}^{\infty}\frac{P_n(w,c)P_m(w,d)}{(aw,bw;q)_\infty}\,\mathrm{d}\alpha^{(s,t)}(w)$$
$$= \frac{(ac;q)_n(bd;q)_m(abst;q)_\infty}{a^n b^m (as,at,bs,bt;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},as,at;q)_k q^k}{(q,ac,abst;q)_k}\,{}_3\phi_2\left[\begin{matrix} q^{-m},bs,bt\\ bd,abstq^k\end{matrix};q,q\right], \tag{1.9}$$

where polynomial $P_n(a,b) = (a-b)(a-bq)\cdots(a-bq^{n-1})$.

Theorem 2. *For $m,n \in \mathbb{N}$ and $|abst/q| < 1$, we have*

$$\int_{-\infty}^{\infty} P_n(c,w)P_m(d,w)(aw,bw;q)_\infty\,\mathrm{d}\beta^{(s,t)}(w)$$
$$= \frac{c^n d^m (q/(ac);q)_n (q/(bd);q)_m (as,at,bs,bt;q)_\infty}{(abst/q;q)_\infty}$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n},q/(as),q/(at);q)_k}{(q,q/(ac),q^2/(abst))_k}\left(\frac{q^{n+1}}{bc}\right)^k {}_3\phi_2\left[\begin{matrix} q^{-m},q/(bs),q/(bt)\\ q/(bd),q^{2+k}/(abst)\end{matrix};q,\frac{q^{1+m+k}}{ad}\right]. \tag{1.10}$$