J. Math. Anal. Appl. 410 (2014) 455–468

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Optimal control of a singular PDE modeling transient MEMS with control or state constraints

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ARTICLE INFO

Article history:
Received 6 September 2012
Available online 4 September 2013
Submitted by P. Yao

Keywords:
Optimal control of PDEs
Control constraints
State constraints
Singular PDEs
Microelectromechanical systems (MEMS)

ABSTRACT

A particular feature of certain microelectromechanical systems (MEMS) is the appearance of a so-called "pull-in" instability, corresponding to a singularity in the underlying PDE model. We here consider a transient MEMS model and its optimal control via the dielectric properties of the membrane and/or the applied voltage. In contrast to the static case, the control problem suffers from low dimensionality of the control compared to the state and hence requires different techniques for establishing first order optimality conditions. For this purpose, we here use a relaxation approach combined with a localization technique.

1. Introduction

Consider the initial boundary value problem

$$\begin{cases} y_{tt} + cy_t + dy + \rho\Delta^2 y - \eta\Delta y + \dfrac{b(t)a(x)}{(1+y)^2} = 0 & \text{in } Q = (0,T)\times\Omega, \\ y = \partial_\nu y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y = y^0, \qquad y_t = y^1 & \text{in } \{0\}\times\Omega, \end{cases} \tag{1.1}$$

for $\Omega \subseteq \mathbb{R}^n$, $n \in \{1,2,3\}$ (typically $n = 2$), which models the deflection of the membrane of a microelectromechanical system (MEMS), where y is the mechanical displacement, a is the reciprocal of the dielectric coefficient, and b is a dimensionless number proportional to the applied voltage [6]. The constants $c, d \geqslant 0$, $\rho, \eta > 0$ are material parameters, with the term cy_t modeling possible damping and the term dy potentially taking into account the reset force of a spring in the system. The boundary conditions used here correspond to a clamped setting; for a number of different possible boundary conditions we refer to [6]. Fig. 1 shows the schematic of the type of MEMS we are considering here.

The case $y(t,x) = -1$ corresponds to the so-called "pull-in" instability, in which the applied voltage leads to a sufficiently large deflection of the membrane for it to touch the ground plate, possibly damaging the device. This undesirable situation manifests itself in the equation as a potential singularity.

In practical applications, either the dielectric properties, the applied voltage, or both are available as design variables. Consequently, we will consider optimization problems of the form

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.058

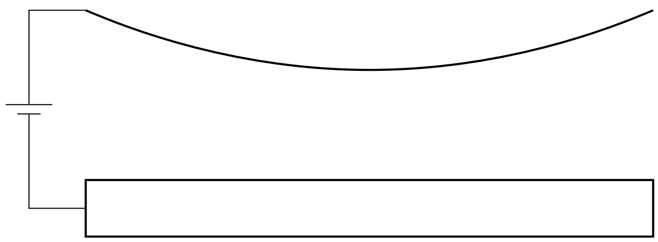

Fig. 1. Schematic of a MEMS.

$$\begin{cases} \min_{y\in\mathcal{Y},u\in\mathcal{U}} \frac{1}{2}\|y-y_d\|^2_{L^2(Q)}+\frac{\alpha}{2}\|u\|^2_{\mathcal{U}} =: J(u,y) \\ \text{s.t.} \quad y_{tt}+cy_t+dy+\rho\Delta^2 y-\eta\Delta y+\frac{\beta u}{(1+y)^2}=0, \\ \quad y|_{\partial\Omega}=\partial_\nu y|_{\partial\Omega}=0, \quad y(0)=y^0, \quad y_t(0)=y^1, \end{cases}$$

with $\mathcal{Y}$ denoting the state space and the control $u\in\mathcal{U}$ being defined by one of the following three cases:

(i) *control by dielectric properties*: $\mathcal{U}=L^2(\Omega)$ and $\beta=b\in L^2(0,T)$ fixed,
(ii) *control by applied voltage*: $\mathcal{U}=L^2(0,T)$ and $\beta=a\in L^2(\Omega)$ fixed,
(iii) *control by both*: $\mathcal{U}$ is a subspace of $L^2(\Omega)\times L^2(0,T)$ (to be defined below) and $\beta\equiv 1$.

For simplicity of exposition and since it is also of high practical relevance, we restrict ourselves to a tracking type cost function for a prescribed target displacement y_d. The existence results below (Theorems 4.1 and 4.6) remain valid for any cost functional $J(u,y)$ that is bounded from below, $\mathcal{U}$-coercive with respect to u (this condition may be omitted in the control constrained case), and weakly lower semi-continuous on $\mathcal{U}$ and $\mathcal{Y}$.

A straightforward approach to prevent instabilities such as the "pull-in" instability at $y=-1$ is to impose control constraints

$$\|u\|_{\mathcal{U}}\leqslant M_u$$

with M_u sufficiently small to indirectly – via the PDE – guarantee that the state never reaches the critical value $y=-1$. However, the singularity can also be prevented by imposing pointwise state constraints

$$-y(t,x)\leqslant M_y<1.$$

As already demonstrated in [7] for the simpler static MEMS model, only the latter approach is able to attain states corresponding to large deflections, which is relevant in applications for achieving a sufficiently large stroke of the device. In the transient situation with state constraints, due to the reduced dimensionality of the control, the approach from [2] used in [7] is not applicable directly any more. We therefore apply the relaxation approach from [3] together with a localization technique as in [5]. Specifically, we introduce a new independent variable in place of the possibly singular nonlinearity and penalize the deviation from the original minimizers. Taking the limit with respect to the penalty parameter in the corresponding optimality conditions yields the optimality system for the original problem.

This work is organized as follows. In Section 2, we provide some necessary results on the (linearized) state and adjoint equations. Section 3 briefly discusses existence of and optimality conditions for each of the three types of controls above in the control constrained case. The corresponding results for the state constrained case, which form the main contribution of this work, are given in Section 4.

In the following, $C_0(\Omega)$ denotes the completion of the space of all continuous functions with compact support in the simply connected domain $\Omega\subseteq\mathbb{R}^n$ with respect to the norm $\|\cdot\|_{C(\Omega)}$, and $\mathcal{M}(\Omega)$ denotes the space of regular Borel measures (which can be identified with the topological dual of $C_0(\Omega)$). Likewise, for a Banach space X with dual X', $\mathcal{M}(0,T;X')$ denotes the dual of $C(0,T;X)$.

2. State equation

We start with a well-posedness result for a linear problem related to (1.1):

$$\begin{cases} y_{tt}+cy_t+dy+\rho\Delta^2 y-\eta\Delta y+w=0 & \text{in } Q, \\ y=\partial_\nu y=0 & \text{on } (0,T)\times\partial\Omega, \\ y=y^0, \quad y_t=y^1 & \text{in } \{0\}\times\Omega. \end{cases} \tag{2.1}$$