J. Math. Anal. Appl. 410 (2014) 492–498

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Two-stream counter-flow heat exchanger equation with time-varying velocities

Jian-Hua Chen [a,b,*]

[a] School of Mathematical and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, PR China
[b] School of Electrical Engineering, Tel Aviv University, Ramat Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 9 January 2013
Available online 28 August 2013
Submitted by D.L. Russell

Keywords:
Counter-flow heat exchanger equation
Boundary control system
Evolution family
State space method

ABSTRACT

We study the two-stream counter-flow heat exchanger equation with time-varying fluid velocities. Formulating it into a time-varying boundary control system, a representation formula of the solution is obtained.

1. Introduction

Heat exchangers are widely used in a large variety of industrial processes and engineering experiments to facilitate the transfer of heat [10,11] between hot and cold fluids or gases. Among which *parallel-flow heat exchangers* and *counter-flow heat exchangers* are most common. Topics related with them have received much attention in recent years, see for instance [1,2,6,7,9,10,14,15,21]. In this paper, we are concerned with the following two-stream counter-flow heat exchanger equation

$$\begin{cases} \dfrac{\partial z_1}{\partial t}(x,t) = -v_1(t)\dfrac{\partial z_1}{\partial x}(x,t) + h_1\big[z_2(x,t) - z_1(x,t)\big], & \\ \dfrac{\partial z_2}{\partial t}(x,t) = v_2(t)\dfrac{\partial z_2}{\partial x}(x,t) + h_2\big[z_1(x,t) - z_2(x,t)\big], & t > 0,\ 0 < x < l, \\ z_1(0,t) = u_1(t), \qquad z_2(l,t) = u_2(t), & t \geqslant 0, \\ z_1(x,0) = z_{10}(x), \qquad z_2(x,0) = z_{20}(x), & 0 < x < l. \end{cases} \tag{1.1}$$

Here $z_1(x,t)$ and $z_2(x,t)$ denote the temperatures of the hot and cold fluids at time t and position x, respectively; $u_1(t)$ and $u_2(t)$ indicate the boundary inputs; $v_1(t)$ and $v_2(t)$ are positive functions reflecting the velocities of the two fluids at time t; h_1, h_2 are positive thermal constants and l stands for the length of the heat exchanger. See Fig. 1 for the sketch of the heat exchanger. The above system has been studied by Grabowski [6] and its derivation was given in [6, Appendix A]. But, instead of (1.1) itself, what Grabowski actually studied is its variation (see (6) in [6]) corresponding to the *equilibrium state*

$$v_1(t) \equiv v_{1\infty} > 0, \qquad v_2(t) \equiv v_{2\infty} > 0, \qquad \frac{\partial z_1}{\partial t} = \frac{\partial z_2}{\partial t} = 0.$$

* Correspondence to: School of Mathematical and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, PR China.
E-mail address: cjhmath@mail.ustc.edu.cn.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.038

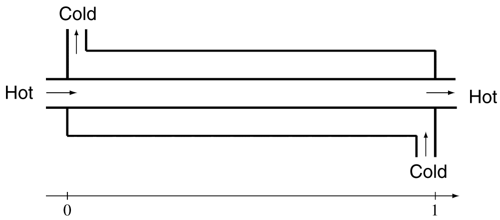

Fig. 1. Sketch of the two-stream counter-flow heat exchanger of unit length. The fluids enter the exchanger from opposite ends.

This leads to a standard autonomous Cauchy system in the state space $Z = L^2(0,l) \times L^2(0,l)$ where the coefficients are no longer time-varying. The control operator is bounded and the unbounded observation operator is defined by the output function

$$y(t) = x_2(0,t),$$

i.e., temperature observation of the cold fluid at $x = 0$. It was shown that the system is *exponentially stable* [6, Lemma 1] and that the observation functional is *infinite-time admissible* [6, Lemma 2]. In addition, using a *circle criterion* coming from the *Leray–Schauder's fixed point theorem*, the author also investigated the closed-loop system with nonlinear feedback $u(t) = -f[y(t)]$, see [6, Theorem 2] for details.

Whereas many numerical methods have been developed (see e.g. [10, Sect. 3]), some researchers aim at finding solution formulas of heat exchanger equations [1]. This is also the main goal of the present paper. Note that in the case when $v_1(t)$ and $v_2(t)$ are time-independent, following discussion in [9] and formulating (1.1) into a *boundary control system* [4,20], a solution formula can be obtained easily.

The general case gives rise to a time-varying boundary control system and it is more complicated. First of all, instead of *strongly continuous semigroups*, we have to deal with *evolution families*, see for instance [18], [13, Definition 5.5.3] and [5, Definition VI.9.2]. As compared with the semigroup case, in general, we have no explicit expressions for the evolution families which adds many technical difficulties. However, as we will see, the underlying evolution family of (1.1) has an explicit expression and what is more, there is a natural generalization of [4, Theorem 3.3.4] to the time-varying case from which a representation formula of the solution follows.

The rest of this paper is organized as follows. In Section 2, we formulate (1.1) into a time-varying boundary control system. In Section 3, a special type of time-varying boundary control systems including the one associated with (1.1) is studied. A representation formula is given for such systems. Section 4 is devoted to conclusions.

2. A time-varying boundary control system

In this section, we formulate (1.1) into a time-varying boundary control system. First, we introduce the state space

$$Z := L^2(0,l) \times L^2(0,l)$$

and the input space $U := \mathbb{R} \times \mathbb{R}$. Next, we define the operator

$$D := \begin{bmatrix} -\frac{d}{dx} & 0 \\ 0 & \frac{d}{dx} \end{bmatrix} \tag{2.1}$$

with domain

$$\mathcal{D}(D) := \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in H^1(0,l) \times H^1(0,l) \colon f_1(0) = f_2(l) = 0 \right\}. \tag{2.2}$$

Denote $S_r(t)$ and $S_l(t)$ the right-shift semigroup and left-shift semigroup on $L^2(0,l)$, namely, for each $f \in L^2(0,l)$,

$$\big(S_r(t)f\big)(x) = \begin{cases} f(x-t), & x \geqslant t, \\ 0, & x < t, \end{cases} \qquad \big(S_l(t)f\big)(x) = \begin{cases} f(x+t), & x+t \leqslant l, \\ 0, & x+t > l. \end{cases}$$

Then it follows that D is the generator of the C_0-semigroup

$$S(t) = \begin{bmatrix} S_r(t) & 0 \\ 0 & S_l(t) \end{bmatrix}. \tag{2.3}$$

By the way, it is clear that both $S_r(t)$ and $S_l(t)$ are *nilpotent*, precisely, $S_r(t) = S_l(t) = 0$ for all $t > l$ which implies $S(t)$ is also nilpotent and $S(t) = 0$ for all $t > l$. Further, writing

$$H := \begin{bmatrix} -h_1 & h_1 \\ h_2 & -h_2 \end{bmatrix}, \qquad V(t) := \begin{bmatrix} v_1(t) & o \\ 0 & v_2(t) \end{bmatrix} \tag{2.4}$$