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Spectral analysis and exponential stability of one-dimensional wave equation with viscoelastic damping ☆

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ABSTRACT

This paper presents the exponential stability of a one-dimensional wave equation with viscoelastic damping. Using the asymptotic analysis technique, we prove that the spectrum of the system operator consists of two parts: the point and continuous spectrum. The continuous spectrum is a set of N points which are the limits of the eigenvalues of the system, and the point spectrum is a set of three classes of eigenvalues: one is a subset of N isolated simple points, the second is approaching to a vertical line which parallels to the imagine axis, and the third class is distributed around the continuous spectrum. Moreover, the Riesz basis property of the generalized eigenfunctions of the system is verified. Consequently, the spectrum-determined growth condition holds true and the exponential stability of the system is then established.

1. Introduction

It is known that the viscoelastic materials have been widely used in mechanics, chemical engineering, architecture, traffic, information and so on [3,16]. Many researchers have paid close attention to the dynamic behavior and control of vibration for elastic structures with viscoelasticity in the past several decades. In the early 1990s, the existence and asymptotic stability of a linear hyperbolic integro-differential equation are presented for the Hilbert state space in [4], where an abstract version of the equation of motion for dynamic linear viscoelastic solids is established. The well-posedness for damped second-order systems with unbounded input operators is considered in [1], and the existence, uniqueness and continuous dependence of solutions in a weak or variational setting are presented. Later on, using a frequency domain method and combining a contradiction argument with the multiplier technique, the exponential stability for a vibrating Euler–Bernoulli beam with Kelvin–Voigt damping distributed locally on any subinterval of the region is studied in [8], and the stability for a vibrating string with local viscoelasticity, that is, one segment of the string is made of viscoelastic material and the other segments are made of elastic material, is discussed in [9]. In [11], the global existence and the asymptotic behavior of the solution to a non-linear one-dimensional wave equation with a viscoelastic boundary condition are analyzed by means of the energy method. Spectral analysis of a wave equation with Kelvin–Voigt damping is considered in [5] and it is shown that, with some assumption of the analyticity of the variable coefficients, the continuous spectrum of the system is an interval on the left real axis in [5]. The Riesz basis property of the generalized eigenfunctions of a one-dimensional hyperbolic system, which describes a heat equation incorporating the effect of thermomechanical coupling and the effect of inertia, is studied in [14]. The mathematical equation modeling a vibrating Timoshenko beam, which is made of viscoelastic material of a Kelvin–Voigt type locally in one segment, is deduced and the exponential stability is obtained under certain

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hypotheses of the smoothness and structural condition of the coefficients of the system in [18]. In [15], a detailed spectral analysis for a heat equation with thermoelastic memory type is presented. The spectrum-determined growth condition and exponential stability are also concluded in [15]. A particular set of functions related to the controllability of the heat equation with memory and finite signal speed, with suitable kernel, is shown to be a Riesz system in [12]. Other studies from different aspects for elastic structures with viscoelasticity can also be found in [2,13,17,19] and the references therein.

In this paper, we are concerned with the following one-dimensional wave equation with viscoelastic damping under the Dirichlet boundary condition: for $0 < x < 1$, $t > 0$,

$$\begin{cases} w_{tt}(x,t) = a^2 w_{xx}(x,t) - \displaystyle\int_0^t \kappa(t-s) w_{xx}(x,s)\,ds - c w_t(x,t), \\ w(0,t) = w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{cases} \tag{1.1}$$

where the kernel is taken for the finite sum of exponential polynomials:

$$\kappa(t) = \sum_{i=1}^{N} a_i e^{-b_i t}, \quad 0 < a_i,\ b_i \in \mathbb{R},\ 1 \leqslant N \in \mathbb{N}. \tag{1.2}$$

Moreover, the following assumptions hold true for the coefficients:

$$0 < b_1 < b_2 < \cdots < b_N < c, \qquad a^2 > \sum_{i=1}^{N} \frac{a_i}{b_i}. \tag{1.3}$$

In [6], a different model for a vibrating wave system with Boltzmann integrals is considered. The spectral properties are analyzed and the Riesz basis for the system is verified. The spectrum-determined growth conditions and the exponential stability are also concluded.

In this paper, with the viscous damping forced into system (1.1)–(1.2), the dynamic behavior of the system is investigated. By introducing some new variables for the exponential polynomial kernel, we set up a time-invariant system and prove the existence of solution, the distribution and structure of the spectrum, and the basis property of the generalized eigenfunctions.

The paper is organized as follows. In Section 2, some new variables are introduced to transform the system into a time-invariant one. The detailed spectral analysis of the newly formulated system is presented in Section 3. By the asymptotic analysis technique, it is shown that the eigenvalues have three classes: one is the set of the simple points $\{-b_i,\ i = 1, 2, \ldots, N\}$, the second approaches a line that is parallel to the imaginary axis, the third is located around the continuous spectral points which contains N isolated points of the complex plane. Moreover, the residual spectrum is shown to be empty and the set of continuous spectrum is exactly characterized. Section 4 is devoted to the Riesz basis generation and the exponential stability of the system.

2. System operator setup

Set

$$y_i(x,t) = a_i \int_0^t e^{-b_i(t-s)} w_x(x,s)\,ds, \quad i = 1, 2, \ldots, N. \tag{2.1}$$

Then y_i satisfies

$$\begin{cases} (y_i)_t(x,t) = a_i w_x(x,t) - b_i y_i(x,t), \\ (y_i)_x(x,t) = a_i \displaystyle\int_0^t e^{-b_i(t-s)} w_{xx}(x,s)\,ds. \end{cases} \tag{2.2}$$

So we can rewrite (1.1)–(1.2) as

$$\begin{cases} w_{tt}(x,t) - \dfrac{\partial}{\partial x}\left[a^2 w_x(x,t) - \displaystyle\sum_{i=1}^{N} y_i(x,t)\right] + c w_t(x,t) = 0, \\ (y_i)_t(x,t) = a_i w_x(x,t) - b_i y_i(x,t), \quad i = 1, 2, \ldots, N, \\ w(0,t) = w(1,t) = 0, \quad t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad y_i(x,0) = 0, \quad i = 1, 2, \ldots, N. \end{cases} \tag{2.3}$$