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A continued fraction approximation of the gamma function

Cristinel Mortici

Valahia University of Târgovişte, Department of Mathematics, Bd. Unirii 18, 130082 Târgovişte, Romania

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ABSTRACT

The aim of this work is to construct a continued fraction approximation of the gamma function. Some inequalities are established.

1. Motivation

There are many situations in science when we are forced to deal with big factorials. Undoubtedly, the most known and most used formula for approximation of the factorial function is the following

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

now known as Stirling's formula. If in probabilities or statistics, such approximation is satisfactory, in pure mathematics other more precise estimates are necessary. In the recent past, the problems of finding increasingly accurate approximations, and establishing new, sharp inequalities for the large factorials, or for its extension gamma function have attracted the attention of many authors, e.g., [1,2]. A slightly more accurate approximation than Stirling is the Burnside formula [3]:

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}},$$

while much better approximations are the following

$$n! \approx \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n \quad \text{(Gosper [4])},$$

or

$$n! \approx \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{n^3+\frac{1}{2}n^2+\frac{1}{8}n+\frac{1}{240}} \quad \text{(Ramanujan [5])}.$$

E-mail address: cmortici@valahia.ro.

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It is true that these formulas can be improved increasingly more, but at the cost of simple shapes loss. Windschitl [6] suggested in 2002 the following approximation formula for computing the gamma function with fair accuracy on calculators with limited program or register memory:

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(n \sinh \frac{1}{n}\right)^{n/2}. \tag{1}$$

Nemes [7] proposed in 2007 an approximation which gives the same number of exact digits as the Windschitl approximation but is much simpler:

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{12n^2-\frac{1}{10}}\right)^n. \tag{2}$$

These formulas (1)–(2) are stronger than Gosper or Ramanujan formulas, that were considered some of the good approximations having a simple form.

Many authors were preoccupied in the recent past to give increasingly better estimates for gamma and related functions using continued fractions (see, e.g., [8–10,1,11]). As example, in the papers [12,13], [14, pp. 520–521] we find Stieltjes' continued fraction

$$\Gamma(x+1) \sim \sqrt{2\pi x}\cdot\left(\frac{x}{e}\right)^x \exp\left(\frac{a_0}{x+\frac{a_1}{x+\frac{a_2}{x+\cdots}}}\right) \tag{3}$$

where

$$a_0 = \frac{1}{12}, \qquad a_1 = \frac{1}{30}, \qquad a_2 = \frac{53}{210} \text{ etc.}$$

A more complete review of the subject of the gamma function and continued fractions can be obtained from [15,16] and references.

We introduce a new continued fraction approximation starting from Nemes formula (2). Indeed, the main goal of this work is to give a systematic way to construct the following continued fraction approximation for the gamma function

$$\Gamma(x+1) \approx \sqrt{2\pi x}\cdot e^{-x}\left(x+\cfrac{1}{12x-\cfrac{1}{10x+\cfrac{a}{x+\cfrac{b}{x+\cfrac{c}{x+\cfrac{d}{x+\ddots}}}}}}\right)^x, \tag{4}$$

where $a = -\frac{2369}{252}$, $b = \frac{2\,117\,009}{1\,193\,976}$, $c = \frac{393\,032\,191\,511}{1\,324\,011\,300\,744}$, $d = \frac{33\,265\,896\,164\,277\,124\,002\,451}{14\,278\,024\,104\,089\,641\,878\,840}\cdots$ The successive truncations of (4) provides increasingly better approximations, that are more accurate than other classical estimates.

In order to illustrate the technique for deriving (4), we prove first that the best approximation of type

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{12n^2-s}\right)^n, \quad s \in \mathbb{R},$$

is obtained indeed for $s = \frac{1}{10}$. Then we introduce the new approximation

$$n! \approx \sqrt{2\pi n}\cdot e^{-n}\left(n+\cfrac{1}{12n-\cfrac{1}{10n-\frac{2369}{252n}}}\right)^n, \tag{5}$$

improving much the approximations (1)–(2). As the value $s = \frac{1}{10}$, the next coefficient $-\frac{2369}{252}$ is found using the following result.

Lemma 1. *If $(x_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(x_n - x_{n+1}) = l \in \mathbb{R}, \tag{6}$$

with $k > 1$, then there exists the limit:

$$\lim_{n\to\infty} n^{k-1}x_n = \frac{l}{k-1}.$$

This lemma was proven to be useful in [17,2] for constructing asymptotic expansions, or accelerating some convergences. For proof, see, e.g., [17].