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Averaged alternating reflections in geodesic spaces

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ABSTRACT

We study the nonexpansivity of reflection mappings in geodesic spaces and apply our findings to the averaged alternating reflection algorithm employed in solving the convex feasibility problem for two sets in a nonlinear context. We show that weak convergence results from Hilbert spaces find natural counterparts in spaces of constant curvature. Moreover, in this particular setting, one obtains strong convergence.

1. Introduction

The convex feasibility problem for two sets consists of finding a point in the intersection of two nonempty closed and convex sets provided such a point exists. This problem finds remarkable applications in applied mathematics and various branches of engineering (see, for example, [3,11,28,10]) which have motivated many researchers to focus on methods of solving this problem. In Hilbert spaces there exists a wide range of algorithms that use metric projections on the sets in order to obtain sequences of points that converge weakly or in norm (under more restrictive conditions) to a solution of this problem. One of the most famous algorithms is the alternating projection method which was developed by von Neumann [30] and was recently adapted to the setting of CAT(0) spaces by Bačák, Searston and Sims in [2].

Another class of algorithms considered in this respect is based on reflections instead of projections. Given a nonempty closed and convex subset A of a Hilbert space H, the reflection of a point $x \in H$ with respect to A is the image of x by the reflection mapping $R_A = 2P_A - I$, where P_A stands for the metric projection onto A and I is the identity mapping. In this work we focus on the averaged alternating reflection (AAR) method employed in solving the convex feasibility problem for two sets. Suppose A and B are two nonempty closed and convex subsets of a Hilbert space with nonempty intersection. The AAR method generates the following sequence for a starting point $x_0 \in H$: $x_n = T^n x_0$, where $T = \frac{I+R_A R_B}{2}$. This algorithmic scheme was studied by Bauschke, Combettes and Luke in [4,5] not only in connection with the convex feasibility problem, but also for finding a best approximation pair of the sets A and B in case their intersection is empty and such a pair exists. The AAR method was later modified in [6] in order to solve the problem of finding the projection of a point onto the intersection of two closed and convex sets. In fact, for the convex feasibility problem, this algorithm is a special case of one described by Lions and Mercier in [25]. One obtains weak convergence of the sequence (x_n) to a fixed point of the mapping T and the projection of this point onto the set B lies in the intersection of A and B.

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Here we are interested in studying the AAR method in geodesic spaces. However, several difficulties arise when considering this algorithm in a nonlinear setting. First of all one needs to find an appropriate definition for the reflection mapping. In order to guarantee the existence of the reflection of a point in the space, we consider spaces with the geodesic extension property. Moreover, the reflection mapping is not always unique. A second difficulty consists in guaranteeing certain properties of this mapping. In Hilbert spaces, the proof of the convergence of the AAR method relies on the nonexpansivity of the reflection mapping which yields the firm nonexpansivity of the mapping T.

In this paper we prove that the reflection mapping is nonexpansive in spaces of constant curvature and justify why it fails to be nonexpansive in the broad setting of CAT(0) spaces. We also analyze the behavior of reflection mappings in slightly more general settings, namely gluings of model spaces. Likewise, we study the convergence of the AAR method in spaces of constant curvature proving strong convergence in this case. Furthermore, we include a rate of asymptotic regularity for the AAR method.

This work is partly motivated by a communication of Ian Searston given during the 10th International Conference on Fixed Point Theory and its Applications where the problem of studying the nonexpansivity of the reflection mapping in geodesic spaces was raised.

2. Preliminaries

A metric space (X, d) is said to be a *(uniquely) geodesic space* if every two points x and y of X are joined by a (unique) geodesic, i.e., a map $c: [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The image $c([0, l])$ of a geodesic forms a *geodesic segment* which joins x and y and is not necessarily unique. If no confusion arises, we use $[x, y]$ to denote a geodesic segment joining x and y. A point z in X belongs to a geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$ and we write $z = (1 - t)x + ty$ for simplicity. Notice that this point may not be unique. A subset of X is said to be convex if it contains any geodesic segment that joins every two points of it. A *geodesic triangle* $\triangle(x, y, z)$ consists of three points $x, y, z \in X$ (the vertices of $\triangle$) and three geodesic segments joining each pair of vertices (the *edges* of $\triangle$). A *geodesic line* in X is a subset of X isometric to $\mathbb{R}$. A geodesic space has the *geodesic extension property* if each geodesic segment is contained in a geodesic line. More on geodesic metric spaces can be found for instance in [7,27].

The *metric* $d: X \times X \to \mathbb{R}$ is said to be *convex* if for any $x, y, z \in X$ one has

$$d(x, (1 - t)y + tz) \le (1 - t)d(x, y) + td(x, z) \quad \text{for all } t \in [0, 1].$$

A geodesic space (X, d) is *Busemann convex* (introduced in [9]) if given any pair of geodesics $c_1 : [0, l_1] \to X$ and $c_2 : [0, l_2] \to X$ one has

$$d(c_1(tl_1), c_2(tl_2)) \le (1 - t)d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)) \quad \text{for all } t \in [0, 1].$$

It is well known that Busemann convex spaces are uniquely geodesic and with a convex metric.

A very important class of geodesic metric spaces are CAT(k) spaces (where $k \in \mathbb{R}$), that is, metric spaces of curvature uniformly bounded above by k in the sense of Gromov. CAT(k) spaces are defined in terms of comparisons with the model spaces M_k^n, which are the complete, simply connected, Riemannian n-manifolds of constant sectional curvature k. Since these model spaces are of essential importance in this work we give their definition directly as metric spaces and recall some of their properties. For a thorough treatment of such spaces and related topics the reader can check [7,17].

The *n-dimensional sphere* $\mathbb{S}^n$ is the set $\left\{x \in \mathbb{R}^{n+1} : (x \mid x) = 1\right\}$, where $(\cdot \mid \cdot)$ is the Euclidean scalar product. Define $d : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ by assigning to each $(x, y) \in \mathbb{S}^n \times \mathbb{S}^n$ the unique number $d(x, y) \in [0, \pi]$ such that $\cos d(x, y) = (x \mid y)$. Then $(\mathbb{S}^n, d)$ is a metric space called the spherical space. This is a geodesic space and if $d(x, y) < \pi$ then there is a unique geodesic joining x and y. Also, balls of radius smaller than $\pi/2$ are convex. The *spherical law of cosines* states that in a spherical triangle with vertices $x, y, z \in \mathbb{S}^n$ and γ the spherical angle between the geodesic segments $[x, y]$ and $[x, z]$ we have

$$\cos d(y, z) = \cos d(x, y) \cos d(x, z) + \sin d(x, y) \sin d(x, z) \cos \gamma.$$

For $u, v \in \mathbb{R}^{n+1}$, consider the quadratic form given by $\langle u \mid v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^n u_i v_i$. The *hyperbolic n-space* $\mathbb{H}^n$ is the set $\{u = (u_1, u_2, \dots, u_{n+1}) \in \mathbb{R}^{n+1} : \langle u \mid u\rangle = -1, u_{n+1} > 0\}$. Then $\mathbb{H}^n$ is a metric space with the hyperbolic distance $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ assigning to each $(x, y) \in \mathbb{H}^n \times \mathbb{H}^n$ the unique number $d(x, y) \ge 0$ such that $\cosh d(x, y) = -\langle x \mid y\rangle$. The hyperbolic space is uniquely geodesic and all its balls are convex. The *hyperbolic law of cosines* states that in a hyperbolic triangle with vertices $x, y, z \in \mathbb{H}^n$ and γ the hyperbolic angle between the geodesic segments $[x, y]$ and $[x, z]$ we have

$$\cosh d(y, z) = \cosh d(x, y) \cosh d(x, z) - \sinh d(x, y) \sinh d(x, z) \cos \gamma.$$

Let $k \in \mathbb{R}$ and $n \in \mathbb{N}$. The classical *model spaces* M_k^n are defined as follows: if $k > 0$, M_k^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the spherical distance with $1/\sqrt{k}$; if $k = 0$, M_0^n is the n-dimensional Euclidean space $\mathbb{E}^n$; and if $k < 0$, M_k^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the hyperbolic distance with $1/\sqrt{-k}$. The model spaces inherit their geometrical properties from the three Riemannian manifolds that define them. Thus, if $k < 0$, M_k^n is uniquely geodesic, balls are convex and we have a counterpart of the hyperbolic law of cosines. If $k > 0$, there is a unique geodesic segment joining $x, y \in M_k^n$ if and only if $d(x, y) < \pi/\sqrt{k}$. Moreover, closed balls of radius smaller than $\pi/\left(2\sqrt{k}\right)$ are convex and we have a counterpart of the spherical law of cosines. We denote the *diameter of* M_k^n by D_k. More precisely, for $k > 0$, $D_k = \pi/\sqrt{k}$ and for $k \le 0$, $D_k = \infty$.