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A pointwise selection principle for maps of several variables via the total joint variation

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ABSTRACT

Given a rectangle in the real Euclidean n-dimensional space and two maps f and g defined on it and taking values in a metric semigroup, we introduce the notion of the total joint variation $\mathrm{TV}(f, g)$ of these maps. This extends similar notions considered by Hildebrandt (1963) [17], Leonov (1998) [18], Chistyakov (2003, 2005) [5,8] and the authors (2010). We prove the following irregular pointwise selection principle in terms of the total joint variation: *if a sequence of maps* $\{f_j\}_{j=1}^{\infty}$ *from the rectangle into a metric semigroup is pointwise precompact and* $\limsup_{j,k\to\infty} \mathrm{TV}(f_j, f_k)$ *is finite, then it admits a pointwise convergent subsequence* (whose limit may be a highly irregular, e.g., everywhere discontinuous, map). This result generalizes some recent pointwise selection principles for real functions and maps of several real variables.

1. Introduction

Pointwise selection principles (PSP) are assertions which state that under certain specified conditions on a sequence (or a family) of functions, their domain and range, the sequence contains a pointwise convergent subsequence. The known PSP can be classified as regular and irregular. Regular PSP usually apply to sequences of regulated functions (i.e., those having finite one-sided limits at each point of the domain) and additionally assert that analytical properties of the pointwise limit of the extracted subsequence are at least as good as those of the members of the sequence (e.g., it belongs to the same functional class of regulated functions). If this is not the case or no information is available about properties of the pointwise limit, the PSP under consideration is termed irregular. Let us illustrate this by examples.

The classical Helly theorem is a regular PSP: *a pointwise bounded sequence of real functions on a closed interval* $[a, b] \subset \mathbb{R}$ *of uniformly bounded variation admits a pointwise convergent subsequence whose pointwise limit is a function of bounded variation*. This theorem, having numerous applications in Analysis [2–4,7,16,17,19,23], has been generalized for functions and maps of one real variable [2,7,10,12,19] and several real variables [1,4,6,13,17,18,20]; see also references in these papers. The above Helly theorem and all enlisted generalizations are based on the Helly theorem for monotone functions (or its counterpart for monotone functions of several variables [4,18]): *a uniformly bounded sequence of real monotone functions on* $[a, b]$ *contains a pointwise convergent subsequence whose pointwise limit is a bounded monotone function*. Thus, the PSP, alluded to above, are regular.

A different kind of a PSP has been presented in [24]. Given a real function f on $[a, b]$, we denote by $T(f)$ the supremum of sums of the form $\sum_{i=1}^{n} |f(t_i)|$ taken over all $n \in \mathbb{N}$ and all finite collections of points $\{t_1, t_2, \ldots, t_n\} \subset [a, b]$ such that either $(-1)^i f(t_i) > 0$ for all $i = 1, 2, \ldots, n$, or $(-1)^i f(t_i) < 0$ for all $i = 1, 2, \ldots, n$, or $(-1)^i f(t_i) = 0$ for all $i = 1, 2, \ldots, n$ (if f is

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nonnegative on $[a,b]$ or nonpositive on $[a,b]$, we set $T(f)=\sup_{t\in[a,b]}|f(t)|$). The quantity $T(f)$ is called the *oscillation of* f on $[a,b]$. Schrader's generalization of the Helly theorem is as follows: *if a sequence of real functions* $\{f_j\}_{j=1}^{\infty}$ *on* $[a,b]$ *is such that* $\sup_{j,k\in\mathbb{N}} T(f_j-f_k)$ *is finite, then it contains a pointwise convergent subsequence*. In contrast to regular PSP, this result applies to the sequence of non-regulated functions $f_j(t)=(-1)^j\mathcal{D}(t)$, $j\in\mathbb{N}$, $t\in[a,b]$, where $\mathcal{D}$ is the Dirichlet function (which is equal to 1 at rational points and 0 otherwise). Thus, we have an example of an irregular PSP; it is worth noting that it is based on Ramsey's theorem from formal logic (see Theorem A in Section 3). At present even for functions and maps of one real variable only a few irregular PSP are known in the literature [11,12,15], which are, however, more general than PSP based on the Helly theorem for monotone functions.

The purpose of this paper is to present a PSP in the context of maps of several real variables taking values in metric semigroups (i.e., metric spaces equipped with the operation of addition), which, in particular, gives an appropriate framework for treating multifunctions of several variables (cf. [5,7,8,14,22]). In this context a regular PSP has been recently presented in [13] for maps of finite total variation in the sense of Vitali, Hardy and Krause. This paper addresses an irregular PSP, which is expressed in terms of the finite total joint variation and, due to the chosen context, it is of different nature as compared to [15,24] and more close to [11–13].

The paper is organized as follows. In Section 2 we present necessary definitions and our main result (Theorem 1). In order to get to its proof as quickly as possible, in Section 3 we collect all main ingredients and auxiliary facts. Section 4 is devoted to the proof of Theorem 1 and Section 5 contains proofs of the auxiliary results exposed in Section 3.

2. Definitions and the main result

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of positive and nonnegative integers, respectively, and $n\in\mathbb{N}$. Given $x,y\in\mathbb{R}^n$, we write $x=(x_1,\dots,x_n)=(x_i: i\in\{1,\dots,n\})$ for the coordinate representation of x, and set $x+y=(x_1+y_1,\dots,x_n+y_n)$, and $x-y$ is defined similarly. The inequality $x<y$ is understood componentwise, i.e., $x_i<y_i$ for all $i\in\{1,\dots,n\}$, and similar meanings apply to $x=y$, $x\le y$, $y\ge x$ and $y>x$. If $x<y$ or $x\le y$, we denote by I_x^y the rectangle $\prod_{i=1}^n[x_i,y_i]=[x_1,y_1]\times\cdots\times[x_n,y_n]$. Elements of the set $\mathbb{N}_0^n$ are as usual said to be *multiindices* and denoted by Greek letters and, given $\theta=(\theta_1,\dots,\theta_n)\in\mathbb{N}_0^n$ and $x\in\mathbb{R}^n$, we set $|\theta|=\theta_1+\cdots+\theta_n$ (the order of θ) and $\theta x=(\theta_1x_1,\dots,\theta_nx_n)$. The n-dimensional zero $0_n=(0,\dots,0)$ and unit $1_n=(1,\dots,1)$ will be denoted by 0 and 1, respectively (the dimension of 0 and 1 will be clear from the context). We also put $\mathcal{E}(n)=\{\theta\in\mathbb{N}_0^n:\theta\le 1 \text{ and } |\theta| \text{ is even}\}$ (the set of 'even' multiindices) and $\mathcal{O}(n)=\{\theta\in\mathbb{N}_0^n:\theta\le 1 \text{ and } |\theta| \text{ is odd}\}$ (the set of 'odd' multiindices). For elements from the set $\mathcal{A}(n)=\{\alpha\in\mathbb{N}_0^n: 0\ne\alpha\le 1\}$ we simply write $0\ne\alpha\le 1$.

The domain of (almost) all maps under consideration is a rectangle I_a^b with fixed $a,b\in\mathbb{R}^n$, $a<b$, called the *basic rectangle*. The range of maps is a *metric semigroup* $(M,d,+)$, i.e., (M,d) is a metric space, $(M,+)$ is an Abelian semigroup with the operation of addition $+$, and d is translation invariant: $d(u,v)=d(u+w,v+w)$ for all $u,v,w\in M$. A nontrivial example of a metric semigroup is as follows [14,22]. Let $(X,\|\cdot\|)$ be a real normed space and M be the family of all nonempty closed bounded convex subsets of X equipped with the Hausdorff metric d given by $d(U,V)=\max\{\mathrm{e}(U,V),\mathrm{e}(V,U)\}$, where $U,V\in M$ and $\mathrm{e}(U,V)=\sup_{u\in U}\inf_{v\in V}\|u-v\|$. Given $U,V\in M$, defining $U\oplus V$ as the closure in X of the Minkowski sum $\{u+v: u\in U, v\in V\}$, we find that the triple $(M,d,\oplus)$ is a metric semigroup.

Note at once that if $(M,d,+)$ is a metric semigroup, then, by virtue of the triangle inequality for d and the translation invariance of d, we have:

$$d(u+u',v+v')\le d(u,v)+d(u',v'), \tag{2.1}$$

$$d(u,v)\le d(u+u',v+v')+d(u',v'), \tag{2.2}$$

for all $u,v,u',v'\in M$. Inequality (2.1) implies the continuity of the addition operation $(u,v)\mapsto u+v$ as a map from $M\times M$ into M.

Given two maps $f,g: I_a^b\to(M,d,+)$ and $x,y\in I_a^b$ with $x\le y$, we define the *Vitali-type* n-th *joint mixed 'difference' of* f *and* g on $I_x^y\subset I_a^b$ by

$$\begin{aligned}\mathrm{md}_n(f,g,I_x^y)=d\Bigg(&\sum_{\theta\in\mathcal{E}(n)} f\big(x+\theta(y-x)\big)+\sum_{\eta\in\mathcal{O}(n)} g\big(x+\eta(y-x)\big),\\ &\sum_{\eta\in\mathcal{O}(n)} f\big(x+\eta(y-x)\big)+\sum_{\theta\in\mathcal{E}(n)} g\big(x+\theta(y-x)\big)\Bigg).\end{aligned} \tag{2.3}$$

As an example, let us exhibit the form of $\mathrm{md}_n(f,g,I_x^y)$ for the first three dimensions $n=1,2,3$. Since $x=(x_1,\dots,x_n)\in I_a^b$, and likewise for $y\in I_a^b$ and $\theta\in\mathbb{N}_0^n$, we note that the i-th coordinate $x_i+\theta_i(y_i-x_i)$ of $x+\theta(y-x)$ is equal to x_i if $\theta_i=0$ and it is equal to y_i if $\theta_i=1$. Thus, for $n=1$ we have $\mathcal{E}(1)=\{0\}$ and $\mathcal{O}(1)=\{1\}$, and so, $\mathrm{md}_1(f,g,I_x^y)=d(f(x)+g(y),f(y)+g(x))$. If $n=2$, then $\mathcal{E}(2)=\{(0,0),(1,1)\}$ and $\mathcal{O}(2)=\{(0,1),(1,0)\}$, and so,

$$\mathrm{md}_2(f,g,I_{x_1,x_2}^{y_1,y_2})=d\big(f(x_1,x_2)+f(y_1,y_2)+g(x_1,y_2)+g(y_1,x_2),f(x_1,y_2)+f(y_1,x_2)+g(x_1,x_2)+g(y_1,y_2)\big).$$