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Finding eigenvalues for heptadiagonal symmetric Toeplitz matrices

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ABSTRACT

In this paper, a formula for the determinant of heptadiagonal symmetric Toeplitz matrices is obtained. This formula and rational functions are used for studying eigenvalue localization. This work is done by Chebyshev polynomials of the first, second, third and fourth kinds.

1. Introduction

Toeplitz matrices are frequently used in many branches of science and engineering. Banded Toeplitz matrices are an important and large subclass of Toeplitz matrices; see [1,8].

Let f be a Laurent polynomial of the form

$$f(t) = a + bt + bt^{-1} + ct^2 + ct^{-2} + dt^3 + dt^{-3} \tag{1}$$

so

$$f\left(e^{ix}\right) = a + 2b\cos x + 2c\cos 2x + 2d\cos 3x \quad a, b, c, d \in \mathbb{R},\ d \neq 0. \tag{2}$$

The $n \times n$ Toeplitz matrix $T_n(f)$ generated by the function f in L^1 on the complex unit circle T is defined by $T_n(f) = \left(f_{j-k}\right)_{1\le j,k\le n}$ where f_k is the kth Fourier coefficient of f,

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{ix}\right)e^{-ikx}dx. \tag{3}$$

Set

$$\mathbf{P_n} = T_n(f) = \begin{pmatrix} a & b & c & d & & & & & \\ b & a & b & c & d & & & & \\ c & b & a & b & c & d & & & \\ d & c & b & a & b & c & d & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & d & c & b & a & b & c & d \\ & & & d & c & b & a & b & c \\ & & & & d & c & b & a & b \\ & & & & & d & c & b & a \end{pmatrix} \in M_n(\mathbb{R}). \tag{4}$$

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In [4], Elouafi introduced a process for finding eigenvalues of pentadiagonal symmetric Toeplitz matrices. We generalized this process for finding eigenvalues of heptadiagonal symmetric Toeplitz matrices.

We will describe the eigenvalues of the matrix $T_n(f)$ as the zeros of rational functions whose poles and residues are determined explicitly.

For this work we use Chebyshev polynomials of the first, second, third and fourth kinds. Chebyshev polynomials will be denoted by T_n, U_n, V_n and W_n respectively [7].

2. Determinant of the matrix $\mathbf{P}_n$

Let ξ denote any root of $dt^6+ct^5+bt^4+at^3+bt^2+ct+d$, since our polynomial is symmetric so $\frac{1}{\xi}$ is another root. Let $\alpha=\frac{1}{2}(\xi+1/\xi)$ then we have:

$$8d\alpha^3+4c\alpha^2+2(b-3d)\alpha+(a-2c)=0. \tag{5}$$

So Eq. (5) has the three complex roots x, y, z such that

$$x+y+z=-\frac{c}{2d}, \qquad xy+yz+xz=\frac{b-3d}{4d}, \qquad xyz=-\frac{a-2c}{8d} \tag{6}$$

see [6], assume that these roots are distinct and $\xi\neq\pm1$.

Now we begin by recalling some basic properties of the Chebyshev polynomials T_n, U_n, V_n and W_n:

$$\begin{aligned}
&T_0(t)=1, \qquad T_1(t)=t, \qquad T_n(\cos\theta)=\cos n\theta, \quad \text{with roots: } x_k=\cos\frac{\left(k-\frac{1}{2}\right)\pi}{n},\\
&U_0(t)=1, \qquad U_1(t)=2t, \qquad U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with roots: } x_k=\cos\frac{k\pi}{n+1},\\
&V_0(t)=1, \qquad V_1(t)=2t-1, \qquad V_n(\cos\theta)=\frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta}, \quad \text{with roots: } x_k=\cos\frac{\left(k-\frac{1}{2}\right)\pi}{n+\frac{1}{2}},\\
&W_0(t)=1, \qquad W_1(t)=2t+1, \qquad W_n(\cos\theta)=\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \quad \text{with roots: } x_k=\cos\frac{k\pi}{n+\frac{1}{2}},
\end{aligned} \tag{7}$$

with $k=1,\dots,n$.

All Chebyshev polynomials, amongst which the U_i's satisfy the three-term recurrence relation:

$$U_{i+1}(t)=2tU_i(t)-U_{i-1}(t) \quad \text{for } i=1,2,\dots \tag{8}$$

and we have:

$$U_i\left(\frac{1}{2}\left(\xi+\frac{1}{\xi}\right)\right)=\frac{\xi^{i+1}-\frac{1}{\xi^{i+1}}}{\xi-\frac{1}{\xi}}, \quad \xi=e^{i\theta}. \tag{9}$$

Lemma 1. *Let ξ denote any root of the polynomial $dt^6+ct^5+bt^4+at^3+bt^2+ct+d$ and assume $d\neq0$. Set $\alpha=\frac{1}{2}(\xi+1/\xi)$ then*

$$\mathbf{P}_n\begin{pmatrix}U_0(\alpha)\\U_1(\alpha)\\U_2(\alpha)\\\vdots\\U_{n-1}(\alpha)\end{pmatrix}=\begin{pmatrix}dU_1(\alpha)+c\\d\\0\\\vdots\\0\\-dU_n(\alpha)\\-dU_{n+1}(\alpha)-cU_n(\alpha)\\-dU_{n+2}(\alpha)-cU_{n+1}(\alpha)-bU_n(\alpha)\end{pmatrix}$$

and

$$\mathbf{P}_n\begin{pmatrix}U_1(\alpha)\\U_2(\alpha)\\U_3(\alpha)\\\vdots\\U_n(\alpha)\end{pmatrix}=\begin{pmatrix}d-b\\-c\\-d\\0\\\vdots\\0\\-dU_{n+1}(\alpha)\\-dU_{n+2}(\alpha)-cU_{n+1}(\alpha)\\-dU_{n+3}-cU_{n+2}(\alpha)-bU_{n+1}(\alpha)\end{pmatrix}$$