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Mixingales on Riesz spaces

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ABSTRACT

A mixingale is a stochastic process which combines properties of martingales and mixing sequences. McLeish introduced the term mixingale at the 4th Conference on Stochastic Processes and Application, at York University, Toronto, 1974, in the context of L^2. In this paper we generalize the concept of a mixingale to the measure-free Riesz space setting (this generalizes all of the L^p, $1 \le p \le \infty$ variants) and prove that a weak law of large numbers holds for Riesz space mixingales. In the process we also generalize the concept of uniform integrability to the Riesz space setting.

1. Introduction

Mixingales were first introduced by D.L. McLeish in [14]. Mixingales are a generalization of martingales and mixing sequences. McLeish defines mixingales using the L^2-norm. In [14] McLeish proves invariance principles under strong mixing conditions. In [15] a strong law for large numbers is given using mixingales with restrictions on the mixingale numbers.

In 1988, Donald W.K. Andrews used mixingales to present L^1 and weak laws of large numbers, [2]. Andrews used an analogue of McLeish's mixingale condition to define L^1-mixingales. The L^1-mixingale condition is weaker than McLeish's mixingale condition. Furthermore, Andrews makes no restriction on the decay rate of the mixingale numbers, as was assumed by McLeish. The proofs presented in Andrews are remarkably simple and self-contained. Mixingales have also been considered in a general L^p, $1 \le p < \infty$, by, amongst others de Jong, in [6,7] and more recently by Hu, see [10].

In this paper we define mixingales in a Riesz space and prove a weak law of large numbers for mixingales in this setting. This generalizes the results in the L^p setting to a measure free setting. In our approach the proofs rely on the order structure of the Riesz spaces which highlights the underlying mechanisms of the theory. This develops on the work of Kuo, Labuschagne, Vardy and Watson, see [11,12,18,19], in formulating the theory of stochastic processes in Riesz spaces. Other closely related generalizations were given by Stoica [16] and Troitsky [17].

In Section 2 we give a summary of the Riesz space concepts needed as well as the essentials of the formulation of stochastic processes in Riesz spaces. Analogous concepts in the classical probability setting can be found in [3]. Mixingales in Riesz spaces are defined in Section 3 and some of their basic properties derived. The main result, the weak law of large number for mixingales, Theorem 4.2, is proved in Section 4 along with a result on the Cesàro summability of martingale difference sequences.

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2. Riesz space preliminaries

In this section we present the essential aspects of Riesz spaces required for this paper, further details can be found in [1] or [20]. The foundations of stochastic processes in Riesz spaces will also given. These will form the framework in which mixingales will be studied in later sections.

A Riesz space, E, is a vector space over $\mathbb{R}$, with an order structure that is compatible with the algebraic structure on it, i.e. if $f, g \in E$ with $f \leq g$ then $f+h \leq g+h$ and $\alpha f \leq \alpha g$ for $h \in E$ and $\alpha \geq 0,\ \alpha \in \mathbb{R}$. A Riesz space, E, is Dedekind complete if every non-empty upwards directed subset of E which is bounded above has a supremum. A Riesz space, E, is Archimedean if for each $u \in E_+ := \{f \in E | f \geq 0\}$, the positive cone of E, the sequence $(nu)_{n\in\mathbb{N}}$ is bounded if and only if $u = 0$. We note that every Dedekind complete Riesz space is Archimedean, [20, p. 63].

We recall from [1, p. 323], for the convenience of the reader, the definition of order convergence of an order bounded net $(f_\alpha)_{\alpha\in\Lambda}$ in a Dedekind complete Riesz space: (f_α) is order convergent if and only if

$$\limsup_{\alpha} f_\alpha = \liminf_{\alpha} f_\alpha .$$

Here

$$\limsup_{\alpha} f_\alpha = \inf\{\sup\{f_\alpha | \alpha \geq \beta\} | \beta \in \Lambda\},$$
$$\liminf_{\alpha} f_\alpha = \sup\{\inf\{f_\alpha | \alpha \geq \beta\} | \beta \in \Lambda\}.$$

Bands and band projections are fundamental to the methods used in our study. A non-empty linear subspace B of E is a band if the following conditions are satisfied:

(i) the order interval $[-|f|, |f|]$ is in B for each $f \in B$;
(ii) for each $D \subset B$ with $\sup D \in E$ we have $\sup D \in B$.

The above definition is equivalent to saying that a band is a solid order closed vector subspace of E. The band generated by a non-empty subset D of E is the intersection of all bands of E containing D, i.e. the minimal band containing D. A principal band is a band generated by a single element. If $e \in E_+$ and the band generated by e is E, then e is called a weak order unit of E and we denote the space of e bounded elements of E by

$$E^e = \{f \in E : |f| \leq ke \text{ for some } k \in \mathbb{R}_+\}.$$

In a Dedekind complete Riesz space with weak order unit every band is a principal band and, for each band B and $u \in E_+$,

$$P_B u := \sup\{v : 0 \leq v \leq u,\ v \in B\}$$

exists. The above map P_B can be extended to E by setting $P_B u = P_B u^+ - P_B u^-$ for $u \in E$. With this extension, P_B is a positive linear projection which commutes with the operations of supremum and infimum in that $P(u \vee v) = Pu \vee Pv$ and $P(u \wedge v) = Pu \wedge Pv$. Moreover $0 \leq P_B u \leq u$ for all $u \in E_+$ and the range of P_B is B.

In order to study stochastic processes on Riesz spaces, we need to recall the definition of a conditional expectation operator on a Riesz space from [11]. As only linear operators between Riesz spaces will be considered, we use the term operator to denote a linear operator between Riesz spaces. Let $T : E \to F$ be an operator where E and F are Riesz spaces. We say that T is a positive operator if T maps the positive cone of E to the positive cone of F, denoted $T \geq 0$.

In this paper we are concerned with order continuous positive operators between Riesz spaces.

Definition 2.1. Let E and F be Riesz spaces and T be a positive operator between E and F. We say that T is order continuous if for each directed set $D \subset E$ with $f \downarrow_{f\in D} 0$ in E we have that $Tf \downarrow_{f\in D} 0$.

Here a set D in E is said to be downwards directed if for $f, g \in D$ there exists $h \in D$ with $h \leq f \wedge g$. In this case we write $D \downarrow$ or $f \downarrow_{f\in D}$. If, in addition, $g = \inf D$ in E, we write $D \downarrow g$ or $f \downarrow_{f\in D} g$.

Note that if T is a positive order continuous operator with $0 \leq S \leq T$ (i.e. $0 \leq Sg \leq Tg$ for all $g \in E$) then S is order continuous. In particular band projections are order continuous.

Definition 2.2. Let E be a Dedekind complete Riesz space with weak order unit, e. We say that T is a conditional expectation operator in E if T is a positive order continuous projection which maps weak order units to weak order units and has range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of E.

If T is a conditional expectation operator on E, as T is a projection it is easy to verify that at least one of the weak order units of E is invariant under T. Various authors have studied stochastic processes and conditional expectation type operators in terms of order (i.e. in Riesz spaces and Banach lattices), see for example [13,16,17].

To access the averaging properties of conditional expectation operators a multiplicative structure is needed. In the Riesz space setting the most natural multiplicative structure is that of an f-algebra. This gives a multiplicative structure that is compatible with the order and additive structures on the space. The space E^e, where e is a weak order unit of E and E is Dedekind complete, has a natural f-algebra structure generated by setting $(Pe)\cdot(Qe) = PQe = (Qe)\cdot(Pe)$ for band