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# Elongating the partial sums of Faber series

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Dedicated to the memory of our close friend and good mathematician Valeri Martirosian (Yerevan State University, Armenia), 1949–2012

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ABSTRACT

In this paper, we establish the equivalence between the overconvergence of a Faber series and the existence of an elongation of its partial sums, whose arithmetic means converge.



## 1. Introduction

### 1.1. Faber polynomials

Let $K$ be a compact, connected set in the complex plane $\mathbb{C}$ which is non-polar, i.e. not a singleton, and suppose that $K^c := \hat{\mathbb{C}} \setminus K$ is connected, where $\hat{\mathbb{C}}$ denotes the one-point compactification of $\mathbb{C}$. Moreover, let $\phi: K^c \to \{w \in \mathbb{C}: |w| > \varrho\}$ be the unique **conformal mapping** satisfying $\phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\phi(z)}{z} = 1$. Hence, $\phi$ has the following form:

$$\phi(z) = z + \alpha_0 + \frac{\alpha_{-1}}{z} + \cdots,$$

and

$$\{\phi(z)\}^n = z^n + \alpha_{n-1}^{(n)} z^{n-1} + \cdots + \alpha_0^{(n)} + \frac{\alpha_{-1}^{(n)}}{z} + \cdots.$$

The $n$-**th Faber polynomial** with respect to $K$ is then given by

$$F_n(z) := z^n + \alpha_{n-1}^{(n)} z^{n-1} + \cdots + \alpha_0^{(n)}.$$

For every $R > \varrho$, the set $C_R := \phi^{-1}(\{w \in \mathbb{C}: |w| = R\})$ is a closed Jordan curve. We denote the inner domain of this curve by $I(C_R)$, and the outer domain by $O(C_R)$. A **Faber series**

$$f(z) := \sum_{n=0}^{\infty} a_n F_n(z) \quad \text{with} \ \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} \quad \text{and} \quad R > \varrho \tag{1}$$

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converges locally uniformly in $I(C_R)$ and diverges in $O(C_R)$. Contrariwise, given a function $f$ which is holomorphic in $I(C_R)$ with $R > \varrho$, then the following representation holds:

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z) \quad \text{with } a_n = \frac{1}{2\pi i} \int_{C_s} \frac{f(z)\phi'(z)}{\{\phi(z)\}^{n+1}} dz, \quad \varrho < s < R.$$

The series converges locally uniformly in $I(C_R)$, and we have $\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{R}$. For further details about Faber polynomials, we refer to [1, Section 14]. For a Faber series $f$ as in (1), our interest is to study the partial sums $\{s_n(z)\}$, i.e.

$$s_n(z) := \sum_{\nu=0}^{n} a_\nu F_\nu(z). \tag{2}$$

From the above results the following lemma is easily deduced.

**Lemma 1.** *Let $f$ be a Faber series as in (1) with its partial sums $\{s_n(z)\}$ as in (2). Then, the following properties hold.*

1. *The sequence $\{s_n(z)\}$ converges locally uniformly in $I(C_R)$.*
2. *For every $z \in O(C_R)$, we have*

$$\limsup_{n\to\infty} |s_n(z)|^{1/n} = \frac{|\phi(z)|}{R} > 1.$$

In the above situation a Faber series $f$ is called *overconvergent* if there exists a set $S \subseteq O(C_R)$ and an increasing sequence $\{p_k\}$, such that $\{s_{p_k}(z)\}$ converges in $S$. In $S$ pointwise, uniform or locally uniform convergence is possible. Examples of overconvergent Faber series are given in [2–4].

### *1.2. Arithmetic means*

For an arbitrary sequence $\{s_n\}$ we denote by

$$\sigma_n := \frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu, \quad n \in \mathbb{N},$$

the sequence of its arithmetic means. If $\{\sigma_n\}$ converges, then $\{\frac{s_n}{n}\}$ tends to 0, as $n \to \infty$, and if $\{\sigma_n\}$ is bounded, then $\{\frac{s_n}{n}\}$ is also bounded.

Moreover, for an arbitrary sequence $\{s_n\}$ and a sequence $m = \{m_n\}$ of natural numbers, we call the sequence

$$\{\tilde{s}_n\}\colon\ \underbrace{s_0, \ldots, s_0}_{m_0-\text{times}}, \underbrace{s_1, \ldots, s_1}_{m_1-\text{times}}, \ldots, \underbrace{s_n, \ldots, s_n}_{m_n-\text{times}}, \ldots$$

the *m-elongation* of $\{s_n\}$. Obviously, $\{s_n\}$ converges if and only if any $m$-elongation $\{\tilde{s}_n\}$ converges.

If we consider a Faber series as in (1), then the sequence

$$\sigma_n(z) = \frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu(z)$$

of its arithmetic means is locally uniformly convergent in $I(C_R)$ to the value $f(z)$ and diverges in every point $z \in O(C_R)$. However, we shall show in this paper that there is a connection between overconvergence of the considered Faber series on a set in $O(C_R)$ and the existence of an $m$-elongation of its partial sums, whose arithmetic means converge on the same set.

## 2. Statement and proof of the main result

The following theorem is our main result.

**Theorem 2.** *Let $f$ be a Faber series as in (1) with its partial sums $\{s_n(z)\}$ as in (2). For a set $S \subseteq O(C_R)$ the following assertions are equivalent.*

(i) *The Faber series $f$ is* ***overconvergent*** *in $S$, i.e. a subsequence $\{s_{p_k}(z)\}$ converges in $S$.*
(ii) *There exists an $m$-**elongation** of the partial sums $s_n(z)$, whose arithmetic means converge in $S$.*

*The convergences in* (i) *and* (ii) *are of the same kind (pointwise, uniformly, locally uniformly, or almost everywhere) and the limits coincide.*