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Weak convergence analysis of the linear implicit Euler method for semilinear stochastic partial differential equations with additive noise[☆]

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ABSTRACT

In this paper, we analyze the weak error of a semi-discretization in time by the linear implicit Euler method for semilinear stochastic partial differential equations (SPDEs) with additive noise. The main result reveals how the weak order depends on the regularity of noise and that the order of weak convergence is twice that of strong convergence. In particular, the linear implicit Euler method for SPDEs driven by trace class noise achieves an almost optimal order $1-\epsilon$ for arbitrarily small $\epsilon > 0$.

1. Introduction

This article deals with weak approximation order of numerical method for semilinear stochastic partial differential equations with additive noise

$$\begin{cases} dX(t) = \Big(AX(t) + F(X(t))\Big)dt + dW^Q(t), \quad 0 \le t \le T, \\ X(0) = x \in H \end{cases} \tag{1.1}$$

in a real separable Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. Here $-A : \mathcal{D}(-A) \subset H \to H$ is a linear, self-adjoint, positive definite operator, whose domain $\mathcal{D}(-A)$ is dense in H and compactly embedded in H. Further we assume that A generates an analytic semigroup $E(t) = e^{tA}, t \ge 0$, and that $F : H \to H$ satisfies the linear growth condition and is twice continuously Fréchet differentiable with bounded derivatives up to order 2. More accurately, there exists a constant L such that for $\forall y \in H$

$$\|F(y)\| \le L(\|y\| + 1), \tag{1.2}$$

$$\big\|F'(y)\big\|_{\mathcal{L}(H)} \le L, \quad \text{and} \quad \big\|F''(y)\big\|_{\mathcal{L}(H\times H;H)} \le L, \tag{1.3}$$

where the operator norms are defined in Section 2. Moreover, assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ and let Q be a bounded, linear, self-adjoint, positive semi-definite operator in H, with eigenvalues $q_i > 0$

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and corresponding eigenfunctions e_i for $i \in \mathbb{N}$. Then it is a classical result that there exists exactly one element $Q^{\frac{1}{2}} \in \mathcal{L}(H)$ nonnegative and self-adjoint such that $Q = Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}}$. Let $\{W^Q\}_{t\geq 0}$ be a standard Wiener process with respect to $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and with the covariance operator Q. According to [1, Chapter 4], W^Q can be represented as

$$W^Q(t) := \sum_{i\in\mathbb{N}} \sqrt{q_i}\beta_i(t)e_i, \quad t \in [0, T], \tag{1.4}$$

where $\{\beta_i(t)\}$, $i \in \{n \in \mathbb{N}, q_n > 0\}$ for $t \in [0, T]$ are independent real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. To guarantee the existence of the mild solution of (1.1) in H, we further assume that for some positive constant C

$$\left\|(-A)^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\right\|_{\mathcal{L}_2(H)} = \left\|Q^{\frac{1}{2}}(-A)^{\frac{\beta-1}{2}}\right\|_{\mathcal{L}_2(H)} \leq C, \quad \text{for some} \quad \beta \in (0, 1]. \tag{1.5}$$

Here the norm of Hilbert–Schmidt operator space $\|\cdot\|_{\mathcal{L}_2(H)}$ is defined in the next section. Under the assumptions above, one can easily verify that the SPDE (1.1) has a unique mild solution $X : [0, T] \times \Omega \to H$ with continuous sample path, given by

$$X(t) = E(t)x + \int_0^t E(t-s)F(X(s))\, ds + \int_0^t E(t-s)dW^Q(s), \quad \mathbb{P}\text{-a.s.}, \tag{1.6}$$

which satisfies

$$\mathbb{E}\|X(t)\|^2 < \infty, \quad t \in [0, T]. \tag{1.7}$$

Note that the condition (1.5) is used to ensure that the stochastic integral in (1.6) is well-defined in H (see [2, Theorem 3.1]). Now we approximate Eq. (1.1) in time by the linear implicit Euler method. Given $2 \leq M \in \mathbb{N}$ and stepsize $\Delta t = \frac{T}{M}$, the linear implicit Euler method is given by $Y_0 = x$ and for $m = 1, 2, \ldots, M$

$$Y_m = E_{\Delta t}Y_{m-1} + \Delta t E_{\Delta t}F(Y_{m-1}) + E_{\Delta t}\Delta W^Q_{m-1}, \tag{1.8}$$

where for simplicity of notation we write $E_{\Delta t} := (I - \Delta tA)^{-1}$ and $\Delta W^Q_{m-1} := W^Q(t_m) - W^Q(t_{m-1})$. We remark that the noise term in (1.8) is also well-defined in H due to (1.5) and (2.11).

For a numerical scheme, various notions of convergence can be taken into account. Two most important notions among them are strong convergence and weak convergence, which are concerned with the pathwise approximation and approximation of the law, respectively. For finite dimensional stochastic differential equations, both strong and weak convergence have been thoroughly investigated, see, e.g., [3,4] and references therein. Compared with the finite dimensional case, numerics of stochastic differential equations in infinite dimensions are much more complicated due to the presence of unbounded operator A. In the past decade, plenty of work has been done on the strong convergence of numerical methods for SPDEs ([2,5–16] and see the review article [17] for more references). On the contrary, just a few literature [18–23] focus on the weak convergence, which is sometimes more interesting in many applications. This work will investigate weak convergence order of semi-discretization in time by the method (1.8) applied to (1.1) and weak convergence of full discretization will be our future work. To be more precise, the aim of this paper is to measure the weak error

$$|\mathbb{E}\varphi(X(T)) - \mathbb{E}\varphi(Y_M)|, \quad \text{as } \Delta t \to 0,$$

where φ is a suitable class of functions. Similarly to the existing work [18–21,23] on weak convergence, we choose the test function space $C_b^2(H; \mathbb{R})$, namely, the set of all real-valued, twice Fréchet differentiable function φ whose first and second derivatives are continuous and bounded. Our main result (Theorem 2.1) covers equations with both space–time white noise and trace class noise. The result indicates that the weak order depends heavily on the regularity of the noise. Particularly, in the case of trace class noise, i.e., $\beta = 1$ and $\mathrm{Tr}(Q) < \infty$, we can almost get the optimal order one for the linear implicit Euler method. Moreover, the result shows that in all cases the rate of weak convergence is twice that of strong convergence.

We mention that, for semilinear SPDEs (1.1) with additive noise, where W^Q is a standard Wiener process including both space–time white noise ($Q = I$) and trace class noise ($\mathrm{Tr}(Q) < \infty$), many authors [9–11] have studied the strong convergence of various numerical schemes. But for weak convergence of numerical methods, only linear stochastic evolution problems with such general additive noise have been considered in [18,20,21]. This article will fill the gap and focus on the weak convergence of the scheme (1.8) for the semilinear SPDEs with general additive noise. For a linear equation with additive noise, whose solution can be written down explicitly, the authors of [18,21] get rid of the term involving the unbounded operator A and further use a change of variable to simplify the proof. But for nonlinear equation (1.1), whose solution cannot be written down explicitly, one cannot generalize the ideas mentioned above since the operator A considered in our work does not generate a group (see the introduction in [19,24]). To address this problem, we directly invoke the Kolmogorov equation and Itô's formula to decompose the weak error into several terms. Further, these terms are estimated by using Malliavin calculus, the Taylor formula in Banach space and some regularity results. It is worthwhile to point out that we follow some ideas in [19] to decompose the weak error and to estimate the resulting terms. In [19], only space–time white noise ($Q = I$) was considered. Due to the presence of $Q^{\frac{1}{2}}$ or Q, however, new techniques are developed here to estimate the terms involving $Q^{\frac{1}{2}}$ or Q (see, for example, the estimate of a_k^3 in Section 4.2 and the estimate of IV in Section 4.4). Moreover, better regularity results (Lemma 3.3) are achieved and more careful calculations (see, e.g., estimate of I in Section 4.1)