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The numerical range of banded biperiodic Toeplitz operators

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ABSTRACT

The numerical range of an operator is a well studied concept with many applications in several areas of mathematics. In this paper, the numerical range of banded biperiodic Toeplitz operators is investigated, performing a reduction to the 2×2 case. Namely, the parametric equations of the boundary generating curves are deduced and two algorithms for the numerical generation of the numerical range are presented.

1. Introduction

Let M_n be the algebra of $n \times n$ complex matrices. A matrix $T_n = (t_{kj}) \in M_n$ is said to be a *biperiodic Toeplitz* matrix if $t_{kj} := a_{k-j}$, for k odd, and $t_{kj} := b_{k-j}$, for k even, $k, j = 1, \ldots, n$. If there exists an integer $m \in \mathbb{N}$, $m < n$, such that $a_{k-j} = 0$ and $b_{k-j} = 0$, for $|k-j| > m$, $k, j = 1, \ldots, n$, then T_n is said to be a *banded biperiodic Toeplitz matrix* with *bandwidth* $2m+1$. Let l^2 be the Hilbert space of complex valued sequences $\{x_n\}_{n=0}^{+\infty}$, such that the series $\sum_{n=0}^{+\infty} |x_n|^2$ converges, endowed with the usual inner product $\langle x, y\rangle = \sum_{k=0}^{+\infty} x_k \bar{y}_k$. An infinite biperiodic Toeplitz matrix T with bandwidth $2m + 1$ is completely determined by its entries in the $(m + 1)$th and $(m + 2)$th rows, that is, by the sequences

$$\begin{aligned} \{t_{m+1,k}\}_{k=1}^{\infty} &= \{a_m, a_{m-1}, \ldots, a_0, a_{-1}, a_{-2}, \ldots, a_{-m}, 0, \ldots\}, \\ \{t_{m+2,k}\}_{k=1}^{\infty} &= \{0, b_m, b_{m-1}, \ldots, b_0, b_{-1}, b_{-2}, \ldots, b_{-m}, 0, \ldots\}, \end{aligned} \tag{1}$$

if m is even, and by

$$\begin{aligned} \{t_{m+1,k}\}_{k=1}^{\infty} &= \{b_m, b_{m-1}, \ldots, b_0, b_{-1}, b_{-2}, \ldots, b_{-m}, 0, \ldots\}, \\ \{t_{m+2,k}\}_{k=1}^{\infty} &= \{0, a_m, a_{m-1}, \ldots, a_0, a_{-1}, a_{-2}, \ldots, a_{-m}, 0, \ldots\}, \end{aligned} \tag{2}$$

if m is odd. The infinite matrix T induces a bounded linear operator $T : l^2 \times l^2 \to l^2 \times l^2$, which acts by the rule $Y = TX$, where X is the vector $\left(\{x_n\}_{n=0}^{+\infty}, \{y_n\}_{n=0}^{+\infty}\right) \in l^2 \times l^2$ written in the form of a column vector $X = [x_0, y_0, x_1, y_1, \ldots]^T$. In the sequel, we use T to represent interchangeably the infinite matrix T and the linear operator induced by T on $l^2 \times l^2$. These operators occur in many problems in mathematics and physics, such as the chain model of electronic structure [1] or the normal mode theory of one dimensional crystal [2].

Our investigation concerns the numerical range of banded biperiodic Toeplitz operators. The *numerical range* of a bounded linear operator A defined on a Hilbert space $\mathcal{H}$ with an inner product $\langle .,.\rangle_{\mathcal{H}}$ is the subset of the complex plane defined as

$$W(A) := \left\{ \frac{\langle Ax, x\rangle_{\mathcal{H}}}{\langle x, x\rangle_{\mathcal{H}}} : x \in \mathcal{H},\ \ \langle x, x\rangle_{\mathcal{H}} \neq 0 \right\}.$$

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The Toeplitz–Hausdorff theorem asserts that $W(A)$ is convex and its closure contains the spectrum of A, $\sigma(A)$. If A is normal, then $\overline{W(A)}$ is the convex hull of $\sigma(A)$, throughout denoted by $\mathrm{Co}\,\sigma(A)$. Further, every extreme point (corner) of $W(A)$ is an eigenvalue of A. If $A \in M_n$ is *unitarily reducible*, that is, $A = U^*(A_1 \oplus \cdots \oplus A_n)U$ for some unitary matrix U and $n \geq 2$, then $W(A) = \mathrm{Co}\{W(A_1) \cup \cdots \cup W(A_n)\}$. The Elliptical Range Theorem states that the numerical range of $A \in M_2$ is an elliptical disk with foci at λ_1 and λ_2, the eigenvalues of A, and minor axis of length $\left(\mathrm{Tr}\,(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\right)^{1/2}$. If $A \in M_n$, then $W(A)$ is the convex hull of a finite number of algebraic curves [3]. Nevertheless, for certain types of matrices the numerical range is still an elliptical disk, independently of the size of the matrices [4,5]. For more details on the basic properties of $W(A)$ see e.g. [6, Chapter 1], [7] or [8].

This paper is organized as follows. In Section 2, asymptotic equivalence of biperiodic circulant and banded biperiodic sequences of Toeplitz matrices is investigated. In Section 3, the numerical range of banded biperiodic Toeplitz operators is characterized, following an approach used in the first algorithm provided in the last section, which relies on the reduction to the 2×2 case. In Section 4, we investigate the numerical range of biperiodic tridiagonal Toeplitz operators, identifying a class with an elliptical range. In Section 5, two algorithms for the generation of the numerical range of banded biperiodic Toeplitz operators are presented.

2. Asymptotic equivalence of biperiodic circulant and banded biperiodic sequences of Toeplitz matrices

In this section we compute the eigenvalues of biperiodic circulant matrices and approximate a sequence of banded biperiodic Toeplitz matrices by an asymptotically equivalent sequence of biperiodic circulant matrices [9]. For this purpose, we introduce the convenient notation and terminology.

A matrix $C_n = (c_{kj})$ such that $c_{kj} = c_{k-j}$, $k, j = 1, \ldots, n$, with $c_k = c_{k-n}$, $k = 1, \ldots, n-1$, is called a *circulant matrix*. Circulant matrices arise in many applications, such as problems involving the discrete Fourier transform [10]. Our study concerns *biperiodic circulant* matrices, that is, matrices $C_n = (c_{kj}) \in M_n$ of even size, such that $c_{kj} = a_{k-j}$, if k is odd, and $c_{kj} = b_{k-j}$, if k is even, $k, j = 1, \ldots, n$, with $a_k = a_{k-n}$, $k = 1, \ldots, n-2$, and $b_k = b_{k-n}$, $k = 2, \ldots, n-1$. We can define a natural embedding of a banded biperiodic Toeplitz matrix T_n with bandwidth $2m+1$ such that $2m+1 < n$, into a biperiodic circulant matrix C_{n+m} (C_{n+m+1}) adding m ($m+1$) rows and m ($m+1$) columns if $m+n$ is even (odd), and filling in the upper right and lower left corners of T_n with appropriate entries. Under these conditions, T_n is the principal submatrix in the first n rows and columns of a biperiodic circulant matrix C_{n+m} (C_{n+m+1}) and we say that T_n is a *compression* of C_{n+m} (C_{n+m+1}). It is clear that, for n even, there is a biperiodic circulant matrix C_n with the same size as T_n such that T_n and C_n only differ in the upper right and lower left corners, for $n > 2m + 1$. For brevity, C_n will be called the biperiodic circulant matrix *associated* with T_n.

Let Γ be the complex unit circle and consider the function $f : \Gamma \to M_2$ defined as follows

$$f(\phi) = T_\phi = \begin{bmatrix} a_e(\phi) & a_o(\phi) \\ b_o(\phi) & b_e(\phi) \end{bmatrix}, \quad 0 \leq \phi < 2\pi, \tag{3}$$

where, for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$a_e(\phi) := \sum_{-m \leq 2k \leq m} a_{2k}\, \mathrm{e}^{ik\phi}, \qquad a_o(\phi) := \sum_{-m \leq 2k-1 \leq m} a_{2k-1}\, \mathrm{e}^{ik\phi} \tag{4}$$

and

$$b_o(\phi) := \sum_{-m \leq 2k+1 \leq m} b_{2k+1}\, \mathrm{e}^{ik\phi}, \qquad b_e(\phi) := \sum_{-m \leq 2k \leq m} b_{2k}\, \mathrm{e}^{ik\phi}. \tag{5}$$

The function f is called the *symbol* of the Toeplitz operator T.

For n even, the $\frac{n}{2}$th complex roots of unity are throughout denoted by

$$\rho_k := \mathrm{e}^{-i\phi_k}, \quad \text{where } \phi_k = \frac{4k\pi}{n},\ k = 0, \ldots, \frac{n}{2} - 1. \tag{6}$$

Theorem 2.1. *For n even, let $C_n \in M_n$ be the biperiodic circulant matrix associated with the biperiodic Toeplitz matrix T_n of bandwidth $2m + 1$. Then there exists a unitary matrix $U \in M_n$ such that*

$$C_n = U\left(T_{\phi_0} \oplus \cdots \oplus T_{\phi_{\frac{n}{2}-1}}\right)U^*,$$

where $T_{\phi_k} \in M_2$, are defined in (3). The eigenvalues of C_n are

$$\lambda_\pm(\phi_k) = \frac{1}{2}\left(a_e(\phi_k) + b_e(\phi_k)\right) \pm \frac{1}{2}\sqrt{\left(a_e(\phi_k) - b_e(\phi_k)\right)^2 + 4a_o(\phi_k)b_o(\phi_k)}, \tag{7}$$