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On controllability of a non-homogeneous elastic string with memory

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ABSTRACT

We are motivated by the problem of control for a non-homogeneous elastic string with memory. We reduce the problem of controllability to a non-standard moment problem. The solution of the latter problem is based on an auxiliary Riesz basis property result for a family of functions quadratically close to the nonharmonic exponentials. This result requires the detailed analysis of an integro-differential equation and is of interest in itself for Function Theory. Controllability of the string implies observability of a dual system.

1. Introduction

The conditions of controllability for the broad class of the linear oscillating structures have been under consideration since the classical papers by Fattorini and Russell (see [1,2], the survey [3], and book [4] for the history of the subject and extended list of references). The method of [1], as well as many other subsequent papers, is based on the properties of exponential families (usually in the space $L^2(0,T)$), the most important of which for Control Theory are minimality and the Riesz basis property.

We say that a mechanical system, e.g. a string, is controllable if, for any initial data by suitable manipulation of the exterior forces, the system goes to the given regime. According to the classical scheme outlines above, the solution of the controllability problem is based on an auxiliary basis property result. The last result is of independent interest and represents the central part of the paper.

In this paper, we study controllability of an oscillating string the material of which has the *memory*. Such a string is described by the equation of the form

$$\rho(x)y_{tt}(x,t) = (Ay)(x,t) + \int_0^t N(t-\tau)(Ay)(x,\tau)d\tau, \quad (x,t) \in (0,l) \times (0,T), \tag{1.1}$$

where A is a symmetric differential operator of the second order and we suppose that a control force is applied to one end of the string. Here and below, $N(\cdot)$ is the memory kernel.

In the short review of the literature below, we discuss the results on both heat and wave equation with memory since, formally speaking, controllability for the heat equation may be studied in a way similar to one for the wave equation. Also, the results on control for heat equations with memory motivated our study. We consider only linear models. Also we slightly change the authors' notations to make them uniform.

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To the best of our knowledge, G. Leugering has been the first author to study controllability for viscoelastic systems (see [5]). His model leads to the initial boundary value problem for the heat equation with a memory term. Since the case of the constant coefficients is considered, Laplace transform is used. Monotonicity of the memory kernel is essential for the proof of the exact controllability of the system in finite time.

The model

$$y_{tt} = N_0 \triangle y + \int_0^t N(t-s) \triangle y(x,s)ds - ay_t \tag{1.2}$$

is considered in [6]. Here $N_0 > 0, N \leq 0,$ and $a \geq 0$. The exponential decay of the solution is proved assuming the memory kernel $N(t) \not\equiv 0$ and decays exponentially.

The reachability problem for the wave equation with memory

$$u_{tt} - u_{xx} + \int_0^t e^{-\eta(t-s)} u_{xx}(x,s)ds = 0 \tag{1.3}$$

is studied in [7]. We emphasize the special type of the memory kernel in (1.3). The goal of the current paper is to prove controllability of a similar system with an arbitrary memory kernel (except smoothness requirement).

In the next group of papers, the memory of the material is not necessarily taken into consideration. Yet, these papers contain some technical ideas which are important to us. The wave equation with the time dependent tension is studied in [8–11]. The proof of Riesz basis property of a family of the time-dependent functions in [8,10] is based on the assumption that the tension varies "slowly enough" with time. This restriction is removed in [11]. To the best of our knowledge, the papers [8,10,11] represent the first attempt to apply the method of moments to equations with time dependent coefficients. The new difficulty in this case is the absence of an explicit representation for a family of functions arising in the moment problem. This fact, which substantially complicates the analysis of controllability, is common to the heat equations with memory.

The papers [12,13] are especially important to us. The one-dimensional heat equation for a material with memory is considered there:

$$y_t(x,t) = ay(x,t) + \int_0^t N(t-\tau) y_{xx}(x,\tau)d\tau. \tag{1.4}$$

Controllability problem is reduced to a moment problem with respect to a family of functions. Techniques developed by L. Pandolfi in [12,13] and further developed in [14,15] allow proving that the aforementioned family forms a Riesz basis in a proper L^2 space, and that allows solving the exact controllability problem.

Similarly to [12–15], we reduce the study of controllability to investigation of the Riesz basis property of the auxiliary functions, which we call "quasi-exponentials". However, there is a serious difference between the models discussed in these papers and our model. (a) In these papers, the transformation

$$y \mapsto e^{\theta t} y \quad \text{with } \theta = -N'(0)/N(0) \tag{1.5}$$

leads to an equation of the same form as (1.4) but with $N'(0) = 0$. The last condition appears to be very helpful for the technical purposes. For the wave equation with memory, the similar simplifying condition has the form $N(0) = 0$ which cannot be achieved by a change of variables preserving the structure of equation. This fact sufficiently complicates the analytic problems we solve in the current paper. That is where the difference between our model and the models considered in [12–15] is essential. (b) We note also that the different techniques we use here results in the less restrictive requirements on the kernel $N(t)$ (see Assumption 2 below).

To our knowledge, the present paper is the first work where the boundary controllability of a general 1d-wave equation with memory is studied. Sharp controllability time is established under mild regularity conditions for the (variable) coefficients of the equation.

The inclusion of the memory term in the model makes it necessary to analyze a non-standard moment problem associated with the solutions to a family of the Cauchy problems for an integro-differential equation. We use diverse methods of Functional Analysis and Asymptotic Analysis to prove that these solutions, quasi-exponentials, possess the Riesz basis property in a proper L^2 space. This property allows us solving the moment problem and is of interest in itself for Function Theory.

The paper is organized as follows. In Section 2 we discuss the statement of the problem and the auxiliary Sturm–Liouville problem. In Section 3 we derive a series representation for the solution and the moment problem with respect to the quasi-exponential functions. Section 4 is the central in the paper. Here, we prove basis properties results for these quasi-exponentials. In Section 5 we solve the moment problem and prove the main theorem about controllability, i.e. give the conditions of exact controllability of the string with memory. We also prove the observability of the dual system.

After this paper was submitted, Pandolfi informed us about his submitted paper [16] devoted to controllability of the model which represents a version of our model (1.1). Specifically, the model in [16] appears from our model if we let $\rho(x) \equiv 1$ and use the simplest second order differential operator $A \equiv d^2/dx^2$ instead of the general second order symmetric differential operator. Our results give also more general description of possible controllability situations depending on the