



Reduction operators of Burgers equation

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ABSTRACT

The solution of the problem on reduction operators and nonclassical reductions of the Burgers equation is systematically treated and completed. A new proof of the theorem on the special “no-go” case of regular reduction operators is presented, and the representation of the coefficients of operators in terms of solutions of the initial equation is constructed for this case. All possible nonclassical reductions of the Burgers equation to single ordinary differential equations are exhaustively described. Any Lie reduction of the Burgers equation proves to be equivalent via the Hopf–Cole transformation to a parameterized family of Lie reductions of the linear heat equation.

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1. Introduction

The second-order evolution equation

$$L[u] := u_t + uu_x + u_{xx} = 0 \quad (1)$$

was proposed by Burgers [1,2] as a one-dimensional turbulence model. Eq. (1) is also applied to model other phenomena in physics, chemistry, mathematical biology, etc. A fairly complete review of properties of the Burgers equation can be found in [3, Chapter 4].

It is well known that Eq. (1) is linearized to the heat equation $v_t + v_{xx} = 0$ using the so-called Hopf–Cole transformation $u = 2v_x/v$ [4, p. 102]. At the same time, due to the importance of the Burgers equation for various applications, the exhaustive study of its properties in the framework of symmetry analysis is still topical.

Lie symmetries of the Burgers equation and some of its generalizations were studied since the 1960s. The maximal Lie invariance algebra $\mathfrak{g}^B$ of Eq. (1) was first computed by Katkov [5] in the course of group classification of differential equations of the general form $u_t + uu_x = (f(u)u_x)_x$. The algebra $\mathfrak{g}^B$ is spanned by the vector fields

$$\begin{aligned} \mathcal{P}_t &= \partial_t, & \mathcal{D} &= 2t\partial_t + x\partial_x - u\partial_u, & \mathcal{K} &= t^2\partial_t + tx\partial_x + (x - ut)\partial_u, \\ \mathcal{P}_x &= \partial_x, & \mathcal{G} &= t\partial_x + \partial_u. \end{aligned}$$

The complete point symmetry group G^B of Eq. (1) consists of the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde{u} = \frac{\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{\alpha\delta - \beta\gamma},$$

where $(\alpha, \beta, \gamma, \delta, \kappa, \mu_0, \mu_1)$ is an arbitrary set of constants defined up to a nonzero multiplier, and $\alpha\delta - \beta\gamma = \kappa^2 > 0$. Up to composition with continuous point symmetries, the group G^B contains the single discrete symmetry $(t, x, u) \rightarrow (t, -x, -u)$.

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Generally, reductions of partial differential equations using their Lie symmetries do not provide sufficiently large families of exact solutions of these equations. The nonclassical method of reduction was proposed in [6] (see also [7]) in order to utilize a wider class of vector fields than Lie symmetries. Later such vector fields were called nonclassical symmetries [8] or conditional symmetries [9–11] or reduction operators [12]. The notion of nonclassical symmetries can be extended in several directions, e.g., to the concept of weak symmetry introduced in [13], which is also closely related to compatibility theory of differential equations and the general method of differential constraints [14–18]. Generalized notions of ansatzes and reductions associated with weak symmetries were intensively discussed, e.g., in [13,19–21], see also references therein.

A reduction operator of Eq. (1) is a vector field of the general form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \quad (2)$$

where the coefficients τ and ξ do not simultaneously vanish, which allows one to construct an ansatz reducing the initial Eq. (1) to an ordinary differential equation. See, e.g., [11] for the general definition of involutive families of reduction operators. Every Lie symmetry operator is a reduction operator. The multiplication by nonvanishing functions of (t, x, u) generates an equivalence relation on the set of reduction operators. The determining equations on coefficients of a reduction operator Q are derived from the conditional invariance criterion [10,11,22]

$$Q_{(2)}L[u] \big|_{\mathcal{L} \cap \mathcal{Q}^{(2)}} = 0. \quad (3)$$

Here $Q_{(2)}$ is the second prolongation of the vector field Q , $\mathcal{L}$ is the manifold in the second-order jet space $J^{(2)}$ that corresponds to the Burgers equation $L[u] = 0$, and $\mathcal{Q}^{(2)}$ is the manifold in the same jet space determined by the invariant surface condition $Q[u] = 0$ jointly with its differential consequences $D_t Q[u] = 0$ and $D_x Q[u] = 0$, $Q[u] = \eta - u_t - \xi u_x$ is the characteristic of the vector field Q , D_t and D_x are the operators of total differentiation with respect to t and x , respectively. In view of the evolution kind of the Burgers equation it is natural to partition the set of its reduction operators into two subsets, singular and regular, depending on whether or not the coefficient τ vanishes [23]. Up to the above equivalence relation, one can assume $(\tau, \xi) = (0, 1)$ and $\tau = 1$ for singular and regular reduction operators of the Burgers equation, respectively.

It is the Burgers equation that was first considered from the nonclassical symmetry point of view after the prominent paper [7]. Namely, in [24,25] the determining equations for regular reduction operators of (1) were derived under the gauge $\tau = 1$ and a few of their particular solutions satisfying the additional constraint $\xi_u = 0$ were constructed. The corresponding results are available in [26]. The determining equations for both regular and singular nonclassical symmetries of (1) were presented in [27]. Therein the regular case was studied in detail under the gauge $\tau = 1$, for which the consideration was shown to be partitioned into three cases, $\xi_u = 0$, $\xi_u = 1$ and $\xi_u = -\frac{1}{2}$. The case $\xi_u = 0$ proved to result merely in nonclassical symmetries which are equivalent to Lie symmetries. (Within the framework of the direct method, the same result was earlier obtained in [28] in terms of the corresponding ansatzes and reductions.) The unique reduction operator $\partial_t + u\partial_x$ satisfying the constraint $\xi_u = 1$ was also found and used for reducing the Burgers equation. For the case $\xi_u = -\frac{1}{2}$ some particular solutions of the determining equations jointly with the corresponding ansatzes and invariant solutions of the Burgers equation were constructed. The above consideration of regular nonclassical symmetries from [27] was extended in [29] with more particular solutions satisfying the constraint $\xi_u = -\frac{1}{2}$. Still wider families of particular solutions of the determining equations in this case were given in [22,30]. In [31] an algorithmic procedure to derive determining equations for nonclassical symmetries was proposed, and the Burgers equation was one of the illustrative examples for application of this procedure.

The system $\mathcal{S}_B$ of determining equations for the case $\xi_u = -\frac{1}{2}$ was not well investigated for a surprisingly long time although the study of the analogous system $\mathcal{S}_h$ for regular reduction operators of the linear heat equation $v_t + v_{xx} = 0$, whose form is very similar to $\mathcal{S}_B$, had already been completed in [9,32]. See also [33] for preliminary results on $\mathcal{S}_h$ and [12,34,35] for further generalizations to (1+1)-dimensional second-order linear evolution equations. The system $\mathcal{S}_B$ was first linearized in [36] in a fashion similar to [32]. Namely, this system was reduced by a differential substitution to the uncoupled system of three copies of the linear heat equation. As shown in [37], the systems $\mathcal{S}_B$ and $\mathcal{S}_h$ as well as the substitutions linearizing them can be interpreted in terms of the matrix Burgers equation and the matrix Hopf–Cole transformation.

The above in fact means that the case $\xi_u = -\frac{1}{2}$ can be referred to as “no-go”. In general, looking for reduction operators in a family of vector fields is said to result in a “no-go” case if the corresponding system of determining equations for coefficients of reduction operators is reduced to a well-determined system whose general solution cannot be represented in a closed form, and, moreover, solving this system is equivalent, in a certain sense, to solving the initial equation.

Singular reduction operators of the Burgers equation were in fact not studied until [38,39], where no-go results of [32] on reduction operators with $\tau = 0$ for the linear heat equation were extended to general evolution equations of order greater than one. These results were treated in [23] within the framework of singular reduction operators.

In this paper we intend to enhance and complete the above results on nonclassical symmetries and reductions of the Burgers equation. In particular, extending methods from [12] we present a new proof on the linearization of the system $\mathcal{S}_B$ and show that solutions of this system are expressed via triples of solutions of the Burgers equation. We first exhaustively describe all possible nonclassical reductions of the Burgers equation to single ordinary differential equations including reductions associated with the no-go case $\tau = 1$ and $\xi_u = -\frac{1}{2}$. A part of the description is the assertion stating that any Lie reduction of the Burgers equation is equivalent via the Hopf–Cole transformation to a parameterized family of Lie reductions of the linear heat equation.

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