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On the properties of nonlinear nonlocal operators arising in neural field models

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ABSTRACT

We study the existence and continuous dependence of stationary solutions of the one-population Wilson–Cowan model on the steepness of the firing rate functions. We investigate the properties of the nonlinear nonlocal operators which arise when formulating the stationary one-population Wilson–Cowan model as a fixed point problem. The theory is used to study the existence and continuous dependence of localized stationary solutions of this model on the steepness of the firing rate functions. The present work generalizes and complements previously obtained results as we relax on the assumptions that the firing rate functions are given by smoothed Heaviside functions.

1. Introduction

The macroscopic dynamics of neural networks is often studied by means of neural field models. Here we consider a neural field model of the Wilson–Cowan type [1–5]

$$\frac{\partial}{\partial t}u(x,t) = -u(x,t) + \int_{\mathbb{R}} \omega(x,y)P(u(y,t))dy, \quad x \in \mathbb{R},\ t > 0. \tag{1.1}$$

Eq. (1.1) describes the dynamics of the spatio-temporal electrical activity in neural tissue in one spatial dimension. Here $u(x, t)$ is interpreted as a local activity of a neural population at the position $x \in \mathbb{R}$ and time $t > 0$. The second term on the right hand side of (1.1) represents the synaptic input where P is a firing rate function. Typically P is a smooth function that has sigmoidal shape (the shape of the logistic function). The spatial strength of the connectivity between the neurons is modeled by means of a connectivity function ω. We refer the reader to [1–5] for more details regarding the relevance of Eq. (1.1) in neural field theory.

The most common 'simplification' of the model consists of replacing the smooth firing rate function by the unit step function. The existence of solutions to a neural field equation with smooth firing rate functions can be studied using methods of classical fixed point theory; see e.g. [6,7]. These methods have been applied to the particular type of neural field model by various authors; see [8–11]. Dealing with the unit step function however leads to the discontinuity in the integral operator involved in (1.1), which makes it impossible to apply the classical theory.

Despite difficulties in mathematical treatment, the mentioned 'simplification' allows to obtain closed form expressions for solutions describing coherent structures like stationary localized solutions (bumps) and traveling fronts [5] as well as to assess the stability of these structures using the Evans function approach [12]. To benefit from both representation of P it is often conjectured that the 'simplified' model reproduces the essential features of the model with smooth P in the steep

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firing rate regimes. While this conjecture is supported by numerical simulations (see for example [13]) there are few and far between works addressing this problem in a rigorous mathematical way. Namely, Potthast and Beim Graben provided a rigorous approach to study global existence of solutions to the Wilson–Cowan type of the model with the smooth firing rate function as well as with the unit step function, [11]. They demonstrated that the latter case requires more restrictions on the choice of a functional space as well as some extra assumptions on ω. In [14–16] the reader can find the analysis of existence and stability of localized stationary solutions (bumps) for a special class of the firing rate functions, the functions that are 'squeezed' between two unit step functions. This class of functions is also referred to as the smoothed Heaviside functions [17]. It has been shown that if the both bump solutions to the model with the unit step functions are stable/unstable then the bump in the framework of the corresponding smoothed Heaviside firing rate function has the same stability property, [14–16]. To the best of our knowledge, no analysis has been done on the passage from a smooth to discontinuous firing rate function in the framework of neural field models.

In the present paper we study the existence and continuous dependence of stationary solutions to (1.1) under the transition from a smooth firing rate function to the unit step function. The stationary solutions of (1.1) are solutions to a fixed point problem. We describe the fixed point problem in terms of a Hammerstein operator that is represented as the superposition of a Nemytskii operator $\mathcal{N} : u \to P(u(x))$ and a linear integral operator. We study properties of the operators when the firing rate function is represented as a one-parameter family of functions that approach the unit step function with the step taking place in $x = \theta$, when the steepness parameter goes to infinity. The main challenge here is to choose function spaces and a suitable topology of the operators convergence that allow the continuous dependence properties of solutions to be fulfilled.

We introduce the notion of the θ-condition, the condition on a function, say u, to have finite number of only simple roots to $u(x) - \theta$; for details see Definition 3.4. We show that the Nemytskii operator in the limit case (when the steepness parameter goes to infinity) preserves continuity if the functions from the operator domain satisfy the θ-condition. We demonstrate that the choice of the norm is crucial here since, e.g., the θ-condition is achieved in $W^{1,\infty}$-norm but not in $W^{1,q}$-norms, $q < \infty$. Our main results are summarized in Theorems 3.14 and 3.15, which we will refer to as the continuous dependence theorem and the existence theorem, respectively. These theorems enable us to show the existence and continuous dependence of bumps on the steepness parameter when it approaches infinity. We provide two examples of assumptions on ω: one is for the inhomogeneous and one is for the homogeneous function ω, to demonstrate the applicability of our results. In particular, in the latter case we prove the existence of bumps in a steep firing rate regime where the firing rate function takes values zero on a ray $(-\infty, \theta)$. We emphasize that this result is more general than results on the existence of bumps obtained in [14–16].

The paper is organized as follows. In Section 2 we explain our notations, prove some useful theorems, and state lemmas from functional analysis, to which we refer in the subsequent sections. In Section 3 we give a detailed description of the model. Next, we study continuity and compactness of the associated operators in Sobolev spaces, formulate and prove the main theorems. In Section 4 we apply the results of Section 3 to prove continuous dependence of spatially localized stationary solutions (bumps) of (1.1) on the steepness of the firing rate function for both inhomogeneous and homogeneous connectivity functions, and show the existence of the bumps in the framework of the homogeneous ω. Section 5 contains conclusions and outlook.

2. Preliminaries

Let B be an open set of a real Banach space $\mathcal{B}$, then $\overline{B}$ denotes the closure of B in $\mathcal{B}$. We use the notation $\deg(A, B, p)$ for the degree defined for an operator $A : \overline{B} \to \mathcal{B}$, and $p \in \mathcal{B}$. We use $\text{ind}(A, B)$ for the topological index of A, [18].

Let $W^{1,q}(\mathbb{R}, \mu)$, $1 \leq q \leq \infty$, denote a Sobolev space which consists of all functions $w \in L^q(\mathbb{R}, \mu)$ such that their generalized derivatives (with respect to the given measure μ) $dw/d\mu = \tilde{w}$ belong to $L^q(\mathbb{R}, \mu)$.

The element $w \in W^{1,q}(\mathbb{R}, \mu)$ then can be represented as

$$w(x) = w(0) + \int_0^x \widetilde{w}(\xi) d\mu(\xi). \tag{2.1}$$

We consider the following two norms in $W^{1,q}(\mathbb{R}, \mu)$

$$\|w\|_1 = \|w\|_{L^q} + \|\widetilde{w}\|_{L^q} \tag{2.2}$$

and

$$\|w\|_2 = |w(0)| + \|\widetilde{w}\|_{L^q} \tag{2.3}$$

where $\| \cdot \|_{L^q}$ is the norm in $L^q(\mathbb{R}, \mu)$, i.e.,

$$\|w\|_{L^q} = \left(\int_{\mathbb{R}} |w(x)|^q d\mu(x) \right)^{1/q}, \quad 1 \leq q < \infty$$

and

$$\|w\|_{L^\infty} = \sup_{x \in \mathbb{R}} |w(x)|.$$