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Conditional demimartingales and related results

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ABSTRACT

In this paper we deal with the classes of $\mathcal{F}$-demimartingales and conditional N-demimartingales and for these new classes of random objects we provide a number of maximal and moment inequalities as well as related asymptotic results.

1. Introduction

Chow and Teicher [1], Majerek et al. [2], Roussas [3] and Prakasa Rao [4] studied the concept of conditionally independent random variables as well as the concept of conditional association and provided several results. They include conditional versions of generalized Borel–Cantelli lemma, generalized Kolmogorov's inequality, generalized Hájek–Rényi inequalities and further related results.

Prakasa Rao [4] provides counterexamples where independent random variables lose their independence under conditioning and dependent random variables become independent under conditioning.

Conditional association is defined in analogy to (unconditional) association. All random variables are defined on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Following Prakasa Rao [4] for simplicity we will use the notation $E^{\mathcal{F}}(X)$ to denote $E[X|\mathcal{F}]$ where $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{A}$. In addition, $Cov^{\mathcal{F}}(X, Y)$ denotes the conditional covariance of X and Y given $\mathcal{F}$, i.e.,

$$Cov^{\mathcal{F}}(X, Y) = E^{\mathcal{F}}(XY) - E^{\mathcal{F}}(X)E^{\mathcal{F}}(Y).$$

Definition 1. A finite collection of random variables $X_1, \dots, X_n$ is said to be $\mathcal{F}$-associated if

$$Cov^{\mathcal{F}}(f(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0,$$

for any real-valued componentwise nondecreasing functions f, g on $\mathbb{R}^n$ such that the covariance is defined. An infinite collection is $\mathcal{F}$-associated if every finite subcollection is $\mathcal{F}$-associated.

Roussas [3] introduced the concept of conditional negative association as follows.

Definition 2. A finite collection of random variables $X_1, \dots, X_n$ is said to be conditionally negatively associated given $\mathcal{F}$ ($\mathcal{F}$-NA) if

$$Cov^{\mathcal{F}}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0 \quad \text{a.s.},$$

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for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and for any real-valued componentwise nondecreasing functions f, g on $\mathbb{R}^{|A|}$ and $\mathbb{R}^{|B|}$ respectively where $|A| = card(A)$ provided that the covariance is defined. An infinite collection is conditionally negatively associated given $\mathcal{F}$ if every finite subcollection is $\mathcal{F}$-NA.

Yuan et al. [5] provide examples where negative association does not imply conditional negative association and vice versa.

Let us recall the definitions of a sequence of demimartingales and a sequence of N-demimartingales. These two concepts are closely related to the concepts of (unconditional) positive association and negative association respectively.

Definition 3. A sequence of L^1 random variables $\{S_n, n \in \mathbb{N}\}$ is called a demimartingale if for all $j = 1, 2, \ldots$

$$E\left[(S_{j+1} - S_j)f(S_1, \ldots, S_j)\right] \geq 0$$

for all real-valued componentwise nondecreasing functions f whenever the expectation is defined. Moreover, if f is assumed to be nonnegative, the sequence $\{S_n, n \in \mathbb{N}\}$ is called a demisubmartingale.

Definition 4. A sequence of L^1 random variables $\{S_n, n \in \mathbb{N}\}$ is called an N-demimartingale if for all $j = 1, 2, \ldots$

$$E\left[(S_{j+1} - S_j)f(S_1, \ldots, S_j)\right] \leq 0$$

for all real-valued componentwise nondecreasing functions f whenever the expectation is defined. Moreover, if f is assumed to be nonnegative, the sequence $\{S_n, n \in \mathbb{N}\}$ is called an N-demisupermartingale.

Since conditioning has an important role in statistics we introduce the concept of $\mathcal{F}$-demi(sub)martingales and conditional N-demi(super)martingales. These two new classes of random variables are related to the concepts of conditional positive association and conditional negative association respectively.

Definition 5. The sequence $\{S_n, n \geq 1\}$ is called an $\mathcal{F}$-demimartingale if for every real-valued componentwise nondecreasing function f and for $j > i$

$$E\left[\left(S_j - S_i\right) f(S_1, \ldots, S_i) | \mathcal{F}\right] \geq 0 \quad \text{a.s.},$$

where $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{A}$. If moreover f is nonnegative then $\{S_n, n \geq 1\}$ is called an $\mathcal{F}$-demisubmartingale.

Motivated by the definition of $\mathcal{F}$-demi(sub)martingales we present the definition of a sequence of conditional N-demi(super)martingales.

Definition 6. The sequence $\{S_n, n \geq 1\}$ is called a conditional N-demimartingale given $\mathcal{F}$ if for every real-valued componentwise nondecreasing function f and for $j > i$

$$E\left[\left(S_j - S_i\right) f(S_1, \ldots, S_i) | \mathcal{F}\right] \leq 0 \quad \text{a.s.},$$

where $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{A}$. If moreover f is nonnegative then $\{S_n, n \geq 1\}$ is called a conditional N-demisupermartingale given $\mathcal{F}$.

The rest of the paper is organized as follows: in Section 2 we present results for $\mathcal{F}$-demimartingales while in Section 3 we provide several maximal and moment inequalities for conditional N-demimartingales. Finally, in Section 4 we provide asymptotic results for both $\mathcal{F}$-demimartingales and conditional N-demimartingales.

2. Maximal inequalities for $\mathcal{F}$-demimartingales

It is clear that a sequence of random variables which is an $\mathcal{F}$-demimartingale is always a demimartingale and if moreover f is nonnegative, then an $\mathcal{F}$-demisubmartingale is always a demisubmartingale. The converse need not always be true as is seen by the following example.

Example 7. We define the random variables X_1 and X_2 such that

$$P(X_1 = 5, X_2 = 7) = \frac{3}{8}, \qquad P(X_1 = 5, X_2 = -7) = 0,$$
$$P(X_1 = -3, X_2 = 7) = \frac{1}{8}, \qquad P(X_1 = -3, X_2 = -7) = \frac{4}{8}.$$

It can easily be checked that $\{X_1, X_2\}$ is a demimartingale. Moreover, we assume that f is a nonnegative function. Notice that, given the event $\{| X_1X_2 |= 21\}$,

$$E\left[(X_2 - X_1)f(X_1) |\ | X_1X_2 |= 21\right] = -\frac{6}{8}f(-3) < 0, \tag{1}$$

i.e., $\{X_1, X_2\}$ is not an $\mathcal{F}$-demisubmartingale where $\mathcal{F}$ is the σ-algebra generated by the event $\{| X_1X_2 |= 21\}$.