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Shuffles of copulas and a new measure of dependence

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ABSTRACT

Using a characterization of Mutual Complete Dependence copulas, we show that, with respect to the Sobolev norm, the MCD copulas can be approximated arbitrarily closed by shuffles of Min. This result is then used to obtain a characterization of generalized shuffles of copulas introduced by Durante et al. in terms of MCD copulas and the $*$-product discovered by Darsow, Nguyen and Olsen. Since any shuffle of a copula is the copula of the corresponding shuffle of the two continuous random variables, we define a new norm which is invariant under shuffling. This norm gives rise to a new measure of dependence which shares many properties with the maximal correlation coefficient, the only measure of dependence that satisfies all of Rényi's postulates.

1. Introduction

Since the copula of two continuous random variables is invariant with respect to strictly increasing transformations of the random variables, copulas are regarded as the functions that capture dependence structure between random variables. For many purposes, independence and monotone dependence have so far been considered two opposite extremes of dependence structure. However, monotone dependence is just a special kind of dependence between two random variables. More general complete dependence happens when functional relationship between continuous random variables are piecewise monotonic, which corresponds to their copula being a shuffle of Min. See [1,2]. Mikusiński et al. [3,4] showed that shuffles of Min are dense in the class of all copulas with respect to the uniform norm. This surprising fact urged the discovery of the (modified) Sobolev norm by Siburg and Stoimenov [2] which is based on the $*$-operation introduced by Darsow et al. [5–8]. They [5,6,8,1,2] showed that continuous random variables X and Y are mutually completely dependent, i.e. their functional relationship is any Borel measurable bijection, if and only if their copula has unit Sobolev-norm.

Darsow et al. [5,6,8] showed that for a real stochastic processes $\{X_t\}$, the validity of the Chapman–Kolmogorov equations is equivalent to the validity of the equations $C_{st} = C_{su} * C_{ut}$ for all $s < u < t$, where C_{st} denotes the copula of X_s and X_t. It is then natural to investigate how dependence levels of A and B are related to that of $A * B$. Aside from Π, M and W, the easiest case is when A and B are mutual complete dependence copulas, namely if $\|A\| = \|B\| = 1$ then $\|A * B\| = 1$. Now, if $\|A\| = 1$ and C is a copula then we prove that $A * C$ coincides with a generalized shuffle of C in the sense of Durante et al. [9]. We also give similar characterizations of shuffles of C and generalized shuffles of Min. These characterizations have advantages of simplicity in calculations because it avoids using induced measures. Note that there are many examples where a shuffle of C, i.e. $A * C$ or $C * A$, does not have the same Sobolev norm as C. However, we show that multiplication by unit norm copulas preserves independence, complete dependence and mutual complete dependence.

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Since left- and right-multiplying a copula $C = C_{X,Y}$ by unit norm copulas amount to "shuffling" or "permuting" X and Y respectively, we introduce a new norm, called the $*$-norm, which is invariant under multiplication by a unit norm copula. Mutual complete dependence copulas still has $*$-norm 1. This invariant property implies that complete dependence copulas also possess unit $*$-norm. Based on the $*$-norm, a new measure of dependence is defined in the same spirit as the definition by Siburg and Stoimenov [2]. It turns out that this new measure of dependence satisfies five out of the seven postulates proposed by Rényi [10]. The only known measure of dependence that satisfies all Rényi's postulates is the maximal correlation coefficient.

This manuscript is structured as follows. We shall summarize related basic properties of copulas, the binary operator $*$, the Sobolev norm and Durante et al.'s definitions of generalized shuffles of Min and copulas in Section 2. A characterization of copulas with unit Sobolev norm is obtained in Section 3. Section 4 describes a characterization of generalized shuffles of copulas in terms of the $*$-product and introduces a new norm as well as its properties. And in Section 5, we define a new measure of dependence and verify that it satisfies most of Rényi's postulates.

2. Basics of copulas

A *bivariate copula* is defined to be a joint distribution function of two random variables with uniform distribution on $[0, 1]$. Every copula C induces a measure μ_C on $[0, 1]^2$ by

$$\mu_C([x, u] \times [y, v]) = C(u, v) - C(u, y) - C(x, v) + C(x, y).$$

The induced measure μ_C is doubly stochastic in the sense that for every Borel set B, $\mu_C([0, 1] \times B) = m(B) = \mu_C(B \times [0, 1])$ where m is Lebesgue measure on $\mathbb{R}$. A fundamental property is that the Fréchet–Hoeffding upper bound $M(x, y) = \min(x, y)$ is a copula of continuous random variables X and Y if and only if Y is almost surely a strictly increasing function of X. At the other extreme, the Fréchet–Hoeffding lower bound $W(x, y) = \max(x + y - 1, 0)$ corresponds to continuous random variables being strictly decreasing function of each other. $\Pi(x, y) = xy$ denotes the independence copula.

Perhaps, the most important property of copulas is given by Sklar's theorem [11] which states that to every joint distribution function H of continuous random variables X and Y with marginal distributions F and G, respectively, there corresponds a unique copula $C = C_{X,Y}$, called the *copula of X and Y* for which

$$H(x, y) = C(F(x), G(y))$$

for all $x, y \in \mathbb{R}$. This means that the copula of (X, Y) captures all dependence structure of the two random variables. X and Y are said to be *mutually completely dependent* if there exists an invertible Borel measurable function f such that $P(Y = f(X)) = 1$. Shuffles of Min were introduced by Mikusiński et al. [3] as examples of copulas of mutually completely dependent random variables. By definition, a *shuffle of Min* is constructed by shuffling (permuting) the support of the Min copula M on n vertical strips subdivided by a partition $0 = a_0 < a_1 < \cdots < a_n = 1$. It is shown [3, Theorems 2.1 & 2.2] that the copula of X and Y is a shuffle of Min if and only if there exists an invertible Borel measurable function f with finitely many discontinuity points such that $P(Y = f(X)) = 1$. In [3], such an f is called a *strongly piecewise monotone function*.

Following [5,6], the binary operation $*$ on the set $\mathcal{C}$ of all bivariate copulas is defined as

$$C * D(x, y) = \int_0^1 \partial_2 C(x, t)\partial_1 D(t, y)\, dt \quad \text{for } x, y \in [0, 1]$$

and the *Sobolev norm* of a copula C is defined by

$$\|C\|^2 = \int_0^1 \int_0^1 |\nabla C(x, y)|^2\ dx\, dy = \int_0^1 \int_0^1 ([\partial_1 C](x, y) + [\partial_2 C](x, y))\ dx\, dy.$$

It is well-known that $(\mathcal{C}, *)$ is a monoid with null element Π and identity M. So a copula C is called *left invertible* (*right invertible*) if there is a copula D for which $D * C = M$ ($C * D = M$). It was shown in [5, Theorem 7.6] and [6, Theorem 4.2] that the $*$-product on $\mathcal{C}$ is jointly continuous with respect to the Sobolev norm but not with respect to the uniform norm. Moreover, they [5,6,1] gave a statistical interpretation of the Sobolev norm of a copula.

Theorem 2.1 ([2, Theorems 4.1–4.3]). *Let C be a bivariate copula of continuous random variables X and Y. Then (1) $\frac{2}{3} \le \|C\|^2 \le 1$; (2) $\|C\|^2 = \frac{2}{3}$ if and only if $C = \Pi$; and (3) The following are equivalent.*

a. $\|C\| = 1$.
b. *C is invertible with respect to $*$.*
c. *For each $x, y \in [0, 1]$, $\partial_1 C(\cdot, y), \partial_2 C(x, \cdot) \in \{0, 1\}$ a.e.*
d. *There exists a Borel measurable bijection h such that $Y = h(X)$ a.e.*

It follows readily that all shuffles of Min have norm 1. Recall also that if C is invertible with respect to $*$, its inverse is the transpose copula C^T, defined as $C^T(x, y) = C(y, x)$.

At least as soon as shuffles of Min were introduced in [3], the idea of simple shuffles of copulas was already apparent. See, e.g., [4, p.111]. In [9], Durante et al. gave a general definition of shuffles of copulas via a characterization of shuffles