J. Math. Anal. Appl. 397 (2013) 142–155

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

The direct electromagnetic scattering problem from an imperfectly conducting cylinder at oblique incidence

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ARTICLE INFO

Article history:
Received 27 December 2011
Available online 25 July 2012
Submitted by Hyeonbae Kang

Keywords:
Electromagnetic scattering
Oblique incidence
Uniqueness
Existence
Lax–Phillips method

ABSTRACT

We consider the scattering of a time-harmonic electromagnetic wave by an imperfectly conducting infinite cylinder at oblique incidence. We assume that the cylinder is embedded in an inhomogeneous medium, but both the cylinder and the medium are uniform along the axis of the cylinder. Since the x components and y components of electric field and magnetic field can be expressed in terms of their z components if the cylinder is parallel to the z axis, we can derive from Maxwell's equations and the Leontovich impedance boundary condition that our scattering problem is modeled as a boundary value problem for their z components with oblique boundary condition. Using Rellich's lemma, the uniqueness of solutions to the boundary value problem is justified. To show the existence of its solution, the *Lax–Phillips method* is used. The key point for that is to prove the solvability of the associated oblique derivative problem in a bounded domain consisting of two boundaries which are the boundary of the cross-section of the cylinder with coupled oblique boundary condition and that of a domain containing this cross-section with Dirichlet boundary condition.

1. Introduction

The scattering of a time-harmonic electromagnetic wave by an obstacle embedded in a homogeneous medium in $\mathbb{R}^3$ has been extensively studied, which is usually modeled by a boundary value problem for Maxwell's equations; see [1–3] for an introduction and overview. In the special case when the obstacle is an imperfectly conducting infinite cylinder parallel to the z axis which we call *impedance cylinder* and is illuminated by a TM or TE polarized plane wave at normal incidence, the corresponding electromagnetic scattering can be expressed in terms of z component of either the electric field or the magnetic field. It satisfies the two-dimensional Helmholtz equation and an impedance boundary condition at the boundary of cross-section of the cylinder [4]. However, if the incident plane wave obliquely acts on the cylinder, we cannot decouple the z components of the electric field and the magnetic field even if the incident wave is a pure TM or TE polarized plane wave. Many researchers studied the numerical methods for solving such a kind of scattering problems; see, for example, [5–11]. Most of them focused on the analytic solutions to the electromagnetic scattering problems from special cylinders at oblique incidence. In particular, if the cross-section of the cylinder is circular, one can express the z components of scattered electric field and magnetic field as infinite series of Hankel functions, and then determine the coefficients from the boundary conditions. For the case where the cross-section of a cylinder is of arbitrary shape, a moment method and a finite element method are designed respectively in [5,7] to generate numerical solutions.

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doi:10.1016/j.jmaa.2012.07.049

Although the numerical results presented in these references seem to be efficient and accurate, the solvability of the direct scattering problem has not been rigorously established yet.

In this paper, we will provide a theoretical analysis of the direct scattering problem with oblique incidence. We assume that the exterior medium is dielectric and its inhomogeneity is contained in a bounded domain. Suppose an electromagnetic plane wave is incident on an impedance cylinder at an angle not equal to $\pi/2$ with respect to the negative z axis. We first show that, based on the Maxwell equations, the z components of both electric field and magnetic field satisfy a second order elliptic system in the exterior of D coupled in lower order terms, where D denotes the cross-section of the cylinder. On the boundary ∂D of D, we deduce a coupled oblique boundary condition for the z components of the electric field and the magnetic field from the Leontovich impedance boundary condition. This boundary condition contains some impedance which is a function on ∂D. Then, we establish the solvability for such an oblique derivative problem. Using Rellich's lemma [2], we justify the uniqueness of the solutions to this boundary value problem. For the existence of solutions to this problem, the Lax–Phillips method [12] is used. In applying this method, we need to prove the solvability of the associated boundary value problem in a bounded domain. The difficulty in proving this result consists in the fact that both the equations for the z components of electric field and magnetic field and their boundary conditions on ∂D are coupled with oblique derivatives.

Since there are so many methods to prove the well-posedness of the direct scattering problem, we will give some discussions on other possible methods in relation to our direct scattering problem. In [13], using a variational approach, the authors analyzed a system of two Helmholtz equation in $\mathbb{R}^2$ coupled only via quasi-periodic transmission conditions on a set of piecewise smooth interfaces. Also, in [14], a kind of general exterior boundary value problems for scalar elliptic equations are studied, but the method used there needs C^∞ regularity for the boundary and coefficients. In addition, some related oblique derivative problems for the Laplace and Helmholtz equations are studied in [15–18] by using the integral equation method. It seems that the methods used in these papers are not easy to be adopted to analyze our system of two coupled general second-order elliptic equations subject to a coupled boundary condition, under a reasonably lower regularity assumption.

The rest of this paper is organized as follows. In Section 2, we give the mathematical formulation of the scattering problem for electromagnetic waves scattered from an impedance cylinder at oblique incidence. The unique solvability of the scattering problem is established in Section 3 by first showing its uniqueness and then proving its existence. For the existence, we will use the Lax–Phillips method. Finally, we give some conclusions in Section 4.

2. Formulation of electromagnetic scattering at oblique incidence

We consider the electromagnetic scattering problem of an obliquely incident plane electromagnetic wave by an impedance cylinder. It is not easy to prove the solvability when we allow that the permittivity and the permeability of the exterior medium are *non-constants*.

We assume that the cylinder is parallel to the z axis, and its cross-section D is of arbitrary shape with C^2 boundary ∂D. The permittivity ϵ and the permeability μ of the medium are assumed to be positive C^2 functions of $x,\ y$ but invariant with z. We further suppose that outside a bounded domain $\Omega_0 \subset \mathbb{R}^2$, $\epsilon(x, y)$ and $\mu(x, y)$ are equal to the constants ϵ_0 and μ_0, respectively. In addition, the medium is assumed to be dielectric.

Let

$$(\mathbf{E}^i, \mathbf{H}^i) = (\mathbf{e}^i(x, y), \mathbf{h}^i(x, y))e^{-i\omega t-i\beta z} \tag{2.1}$$

be the time-harmonic incident plane electromagnetic wave with wave number $\tilde{k} = \omega\sqrt{\epsilon_0\mu_0}$, where $\omega > 0$ is the frequency and β is a constant depending on the incident direction. If we denote by $\theta \neq \pi/2$ the incident angle with respect to the negative z axis, we have $\beta = \omega\sqrt{\mu_0\epsilon_0}\cos\theta$. We also define

$$k^2(\mathbf{x}) = \epsilon(\mathbf{x})\mu(\mathbf{x})\omega^2 - \beta^2, \quad \mathbf{x} \in \mathbb{R}^2,$$

which is assumed to be not zero everywhere. For $\mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega_0}$, we have

$$k^2 = \epsilon_0\mu_0\omega^2 - \beta^2 = \epsilon_0\mu_0\omega^2 \sin^2\theta$$

and set

$$k_0 := \omega\sqrt{\epsilon_0\mu_0}\sin\theta.$$

The field (2.1) is defined everywhere and propagate in the background medium in $\mathbb{R}^3$. When it penetrates into Ω_0 and acts on the cylinder, the scattered field is generated with the same space harmonic variation in z as the incident field. We denote the total field outside the cylinder by

$$(\mathbf{E}, \mathbf{H}) = (\mathbf{e}(x, y), \mathbf{h}(x, y))e^{-i\omega t-i\beta z} := \left(\mathbf{e}^i + \mathbf{e}^s, \mathbf{h}^i + \mathbf{h}^s\right)e^{-i\omega t-i\beta z}. \tag{2.2}$$

On the boundary ∂D, we put the Leontovich impedance boundary condition, that is,

$$(\nu \times \mathbf{E}) \times \nu = \lambda(\nu \times \mathbf{H}), \tag{2.3}$$