J. Math. Anal. Appl. 397 (2013) 182–190

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On the divergence theorem on manifolds

Varayu Boonpogkrong [a,b], Tuan Seng Chew [c,*], Peng Yee Lee [d]

[a] Department of Mathematics and Statistics, Faculty of Science, Prince of Songkla University, Hat Yai, 90112, Thailand
[b] Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd, Bangkok, 10400, Thailand
[c] Department of Mathematics, Faculty of Science, National University of Singapore, 10 Lower Kent Ridge Road, 119076, Singapore
[d] Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, 637616, Singapore

ARTICLE INFO

Article history:
Received 22 December 2010
Available online 24 July 2012
Submitted by H.R. Parks

Keywords:
The H–K integral
Partition of unity
Manifolds
The divergence theorem

ABSTRACT

In this paper, the divergence theorem is proved by the Kurzweil–Henstock approach. The physical definition of the divergence of a vector field is used, instead of the usual definition used in mathematics papers. Sufficient conditions for the existence of the divergence of a vector field on n-dimensional manifolds in $\mathbb{R}^n$ are given. Concepts of strong differentiability are used in the sufficient conditions.

1. Introduction

The divergence theorem for vector fields in $\mathbb{R}^n$ is well-known and can be found in many books on multivariate calculus, where the integral used is Riemann's integral and vector fields are continuously differentiable. Using Lebesgue's integral and weakening the condition on the vector fields have been considered by Bochner, Shapiro, Cohen and others; see [1–3]. In the past thirty years, the divergence theorem under the Henstock–Kurzweil integral has been discussed by Pfeffer, Jurkat, Nonnenmacher, Mawhin, Jarnik, Kurzweil, Macdonald and others; see [4–11].

In mathematics papers and books, the usual definition of the divergence div F of a vector field $F = (F_1, F_2, \ldots, F_n)$ is given by $\sum_{i=1}^{n} \partial F_i/\partial x_i$, whereas in physics papers and books, it is given by

$$(\text{div}F)\,(p) = \lim_{\text{diam}(I)\to 0} \frac{1}{|I|} \int_{\partial I} F \cdot \hat{n} ds,$$

where I is an interval containing the point p with surface ∂I and $\hat{n}$ is the exterior normal to ∂I. Physically, this corresponds to the amount of flux of the vector field F out of I divided by the volume $|I|$. Recently, this physical definition of the divergence has been used by Acker, Macdonald and Hubbard; see [12,13,8]. As mentioned in Macdonald's paper, this physical definition fits hand in glove with the Henstock–Kurzweil integral, which is an integral of Riemann type. The divergence theorem can be easily proved. Its proof is intuitive and natural.

In this paper, we shall give sufficient conditions for the existence of the divergence of a vector field on n-dimensional manifolds in $\mathbb{R}^n$.

* Corresponding author.
E-mail addresses: varayu.b@psu.ac.th (V. Boonpogkrong), matcts@nus.edu.sg (T.S. Chew), pengyee.lee@nie.edu.sg (P.Y. Lee).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.07.042

2. Preliminaries

For any fixed positive integer n, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Let $S \subset \mathbb{R}^n$; the boundary and outer Lebesgue measure of S are denoted by ∂S and $|S|$ respectively. Let $x \in \mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$; the norm $\|x\|$ is defined by $\|x\| = \sum_{i=1}^{n} |x_i|$. Let $\eta > 0$; $B(x, \eta)$ or $B_\eta(x)$ denotes $\{y \mid \|x - y\| < \eta\}$. Let E be a compact interval in $\mathbb{R}^n$. A partition P of E is a finite family of non-overlapping compact intervals $\{I_i\}_{i=1}^{m}$ whose union is E. A division D of E is a finite family of point–interval pairs $\{(x_i, I_i)\}_{i=1}^{m}$ such that $\{I_i\}_{i=1}^{m}$ is a partition of E. Let $\delta(x)$ be a positive function defined on E. A point–interval pair (x, I) is said to be McShane δ-fine if $I \subset B(x, \delta(x))$. Suppose x belongs to I, i.e., $x \in I \subset B(x, \delta(x))$. Then (x, I) is said to be Henstock δ-fine. A division D of E is said to be McShane δ-fine if each point–interval pair in D is McShane δ-fine. Similarly we can define Henstock δ-fine divisions.

Let E be a compact interval in $\mathbb{R}^n$ and $f : E \to \mathbb{R}$. Let $D = \{(x_i, I_i)\}_{i=1}^{m}$ be a δ-fine division (McShane or Henstock) of E. We denote the Riemann sum $\sum_{i=1}^{m} f(x_i)|I_i|$ by $S(f, D, \delta)$. In this paper, a division $D = \{(x_i, I_i)\}_{i=1}^{m}$ will often be written as $D = \{(x, I)\}$, in which (x, I) represents the typical point–interval pair in D. The corresponding Riemann sum will be written as $(D) \sum f(x)|I|$.

Definition 1. Let $f : E \to \mathbb{R}$. Then f is said to be McShane integrable to $A \in \mathbb{R}$ on E if for each $\epsilon > 0$, there exists a positive function δ on E such that whenever $D = \{(x, I)\}$ is a McShane δ-fine division of E, we have

$$|S(f, D, \delta) - A| \le \epsilon.$$

We denote A as $(L) \int_E f$.

Definition 2. In the above Definition 1, if "a McShane δ-fine division" is replaced by "a Henstock δ-fine division", then f is said to be Henstock–Kurzweil integrable on E. We denote A as $(HK) \int_E f$.

It is known that f is McShane integrable on E if and only if f is Lebesgue integrable on E; see [14].

The mapping $F : E \to \mathbb{R}^n$ is called a vector field and F is denoted by $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))$, where F_i, $i = 1, 2, \ldots, n$, are real-valued functions on E.

In this paper, we always assume that F is continuous on E. Using the $(n-1)$-dimensional McShane (Lebesgue) integral, we can define the surface integral $(L_{n-1}) \int_{\partial E} F \cdot \hat{n}$ of F over ∂E in the direction of the exterior normal $\hat{n}$. The divergence $\mathrm{div}F$ of F is defined as follows:

$$(\mathrm{div}F)\ (p) = \lim_{\substack{I \subset B(p,\delta(p)) \\ \delta(p) \to 0}} \frac{1}{|I|} \int_{\partial I} F \cdot \hat{n},$$

where I is an interval containing point p and $|I|$ denotes the volume of I. More precisely, for each $\epsilon > 0$, there exists $\delta(p) > 0$ such that for every interval I with $p \in I \subset B(p, \delta(p))$, we have $|(\mathrm{div}F)\ (p)\ |I| - \int_{\partial I} F \cdot \hat{n}| \le \epsilon |I|$.

If $p \in I \subset B(p, \delta(p))$ in the above definition is replaced by $I \subset B(p, \delta(p))$ and p may not belong to I, then F is said to have strong divergence, denoted by $(\mathrm{div}F)_s$.

3. The divergence theorem in $\mathbb{R}^n$

The following proof is given in [8]. The proof is intuitive and natural.

Theorem 1. *Let $F : E \to \mathbb{R}^n$ be a continuous vector field. Suppose the divergence* $\mathrm{div}F$ *exists on E. Then* $\mathrm{div}F$ *is Henstock–Kurzweil integrable on E and*

$$(HK) \int_E \mathrm{div}F = (L_{n-1}) \int_{\partial E} F \cdot \hat{n}.$$

Proof. Suppose the divergence $\mathrm{div}F$ exists on E. Hence for each $x \in E$ and each $\epsilon > 0$, there exists $\delta(x) > 0$ such that for every interval I with $x \in I \subset B(x, \delta(x))$, we have

$$\left| (\mathrm{div}F)\ (x)\ |I| - \int_{\partial I} F \cdot \hat{n} \right| \le \epsilon |I|.$$

Let $D = \{(x, I)\}$ be a Henstock δ-fine division of E. Then we have

$$\left| (D) \sum \left\{ (\mathrm{div}F)\ (x)\ |I| - \int_{\partial I} F \cdot \hat{n} \right\} \right| \le \epsilon (D) \sum |I|.$$

Therefore

$$\left| (D) \sum (\mathrm{div}F)\ (x)\ |I| - \int_{\partial E} F \cdot \hat{n} \right| \le \epsilon |E|.$$