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Limit cycles for planar semi-quasi-homogeneous polynomial vector fields

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ABSTRACT

This paper is concerned with the limit cycles for planar semi-quasi-homogeneous polynomial systems. We give some explicit criteria for the nonexistence and existence of periodic orbits. A lower bound is given for the maximum number of limit cycles of such a system. The cyclicity and center problems are studied for some subfamilies of semi-quasi-homogeneous polynomial systems. Our results generalize those obtained for polynomial semi-homogeneous systems.

1. Introduction and statement of the main results

A function $f(x, y)$ is called a (p, q)-*quasi-homogeneous function of weighted degree* m if $f(\lambda^p x, \lambda^q y) = \lambda^m f(x, y)$ for all $\lambda \in \mathbb{R}$. Let $P_m(x, y)$ and $Q_n(x, y)$ be (p, q)-quasi-homogeneous polynomials of weighted degree $p - 1 + m$ and $q - 1 + n$, respectively. We say that $X = (P_m(x, y), Q_n(x, y))$ is a *planar* (p, q) *semi-quasi-homogeneous polynomial vector field* if $m \neq n$. The system of differential equations associated to X is

$$\frac{dx}{dt} = P_m(x, y) = \sum_{pi+qj=p+m-1} a_{ij} x^i y^j, \qquad \frac{dy}{dt} = Q_n(x, y) = \sum_{pi+qj=q+n-1} b_{ij} x^i y^j. \tag{1}$$

Here p, q, m are positive integers and $P_m(x, y)$ and $Q_n(x, y)$ are coprime in the ring $\mathbb{R}[x, y]$. To be short we denote this by $(P_m(x, y), Q_n(x, y)) = 1$.

Observe that the above definition is the natural one for the following reason:

(i) When $p = q = 1$, X is called *a semi-homogeneous vector field* [1,2]. That is to say, the above definition coincides with the usual definition of the homogeneous case.
(ii) If $m = n$, then X is called *a* (p, q) *quasi-homogeneous vector field of weighted degree* m, see [3, Chapter 7]. In particular if $p = q = 1$, X is *a homogeneous polynomial vector field of degree* m.
(iii) The finite (resp. infinity) singular points of semi-homogeneous vector fields can be studied by homogeneous blow-up (*resp.* Poincaré compactification) whereas the one of semi-quasi-homogeneous vector fields can be studied by the use of quasi-homogeneous blow-up (*resp.* Poincaré–Lyapunov compactification) [4].

One of the classical problems in the qualitative theory of planar polynomial systems is the study of the number of limit cycles, which is known as the 16th Hilbert problem. There has been a substantial amount of work devoted to solving this problem for homogeneous, quasi-homogeneous or semi-homogeneous polynomial vector fields. Li et al. [5] provided an upper bound for the maximum number of limit cycles bifurcating from the period annulus of any homogeneous and quasi-homogeneous centers, which can be obtained using the Abelian integral method. Gavrilov et al. [6] gave a more general

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results on the limit cycles bifurcated from the periodic orbits of quasi-homogeneous centers. Cima and Llibre [7] investigated the algebraic and topological classification of homogeneous cubic vector fields. In 1997, Cima et al. [8] studied the limit cycles for semi-homogeneous vector fields. They proved that semi-homogeneous systems can exhibit periodic orbits only when nm is odd and there exists such a system with at least $(n+m)/2$ limit cycles. In the papers [1,2] Cairó and Llibre study the phase portraits of semi-homogeneous vector fields with $(m, n) = (1, 2)$ and $(m, n) = (1, 3)$ respectively. Some papers are concerned with the integrability [9,10] and structural stability [11–13] for quasi-homogeneous or semi-homogeneous polynomial vector fields.

It is always possible to decompose a vector field $\mathcal{X}$ in a formal series centered in $0 \in \mathbb{R}^n : \mathcal{X} = \sum_{j \ge k} X_j$, where X_j are (p, q)-quasi-homogeneous vector fields of weighted degree k, k is the first integer such that $X_k \neq 0$ [14]. Therefore in order to get more information on limit cycles of general vector fields $\mathcal{X}$ it is essential to study the geometrical properties of quasi-homogeneous and semi-quasi-homogeneous vector fields. We note that Coll, Gasull and Prohens have investigated planar vector fields defined by the sum of two quasi-homogeneous vector fields [15].

In this paper we study the limit cycles of planar semi-quasi-homogeneous polynomial vector fields, defined in (1). To simplify the statements of the main results, first of all we give the following lemma, which will be proved in Section 2.

Lemma 1. *Suppose $(p, q) = k \ge 2$ in (1), then there exists a unique vector (p', q', m', n') with $(p', q') = 1$ such that system (1) is a (p', q') semi-quasi-homogeneous vector field.*

By Lemma 1, we suppose that the following conventions hold without loss of generality.

Conventions. *We always assume in this paper that*

(i) *p is odd and $(p, q) = 1$, and*
(ii) *$P_m(x, y)$ and $Q_n(x, y)$ are coprime polynomials.*

Our main results are the following four theorems. The first one provides explicit criteria for the nonexistence and existence of periodic orbits, and a lower bound of $N = N(p, q, m, n)$ for the maximum number of limit cycles of system (1).

Theorem 2. *Let $P_m(x, y)$ and $Q_n(x, y)$ be (p, q)-quasi-homogeneous polynomials of degree $p-1+m$ and $q-1+n$, respectively, $m \neq n$. For system (1), the following statements hold:*

(i) *If either $q \nmid (p + m - 1)$ or $p \nmid (q + n - 1)$, then system (1) has no periodic orbit.*
(ii) *If $q|(p + m - 1)$ and $p|(q + n - 1)$, then system (1) is reduced to the following normal form*

$$\frac{dx}{dt} = P_m(x, y) = \sum_{i=0}^{[r_1/p]} a_i x^{iq} y^{r_1 - ip}, \qquad \frac{dy}{dt} = Q_n(x, y) = \sum_{j=0}^{[r_2/q]} b_j x^{r_2 - jq} y^{jp}, \tag{2}$$

where $r_1 = (p + m - 1)/q$, $r_2 = (q + n - 1)/p$, $[r_1/p]$ denotes the integer part of r_1/p.
If system (2) has periodic orbits, then one of the following conditions holds:
 (ii.1) *If both p and q are odd, then m, n, r_1 and r_2 are odd.*
 (ii.2) *If p is odd and q is even, then m and n are even, r_1 and r_2 are odd.*
(iii) *If (ii.1) holds, then there exist $P_m(x, y)$ and $Q_n(x, y)$ such that each of the following assertions holds:*
 (a) *All solutions of (2) are periodic.*
 (b) *No solutions of (2) are periodic.*
 (c) *There are periodic and non-periodic solutions of system (2).*
 Let $N = N(p, q, m, n)$ be the maximum number of limit cycles of system (2). Then $N \ge [([r_1/p]+1)/2]+[([r_2/q]+1)/2]-1$.
(iv) *If (ii.2) holds and system (2) has periodic orbits, then the origin is a center.*

The results in Theorem 2 generalize those obtained for semi-homogeneous polynomial vector fields in [8].

One important problem in the qualitative theory of planar polynomial vector fields is the study of the local phase portrait at the singularities to characterize whether a singular point is a center, which is called the center problem. A singular point O of a real planar analytic vector field $\mathcal{X}$ is called a *center* if there exists a neighborhood U of O such that $U \setminus \{O\}$ is filled with periodic integral curves of the vector field. The center is called *elementary* if the linearization $d\mathcal{X}(O)$ of the vector field is a rotation, otherwise the center is called non-elementary. If the origin is a center of system (2), then it is in general non-elementary. The *cyclicity* is the total number of limit cycles which can emerge from a configuration of trajectories (center point, period annulus, separatrix cycle) under a perturbation.

The center and its cyclicity problem are only solved for some special kind of planar polynomial systems: quadratic systems, cubic systems with homogeneous nonlinearities, and so on. In the following two theorems, we give some more information on the center and cyclicity problems for some subfamily of the semi-quasi-homogeneous system (2).

Theorem 3. *Suppose that $q|(p + m - 1)$, $p|(q + n - 1)$ and the condition (ii.1) in Theorem 2 holds. Consider the system*

$$\frac{dx}{dt} = \sum_{i=0}^{[r_1/p]} a_i x^{iq} y^{r_1 - ip}, \qquad \frac{dy}{dt} = b_0 x^{r_2} \tag{3}$$

with $r_1 > r_2$ and $m > n$. If $a_0 b_0 < 0$, then the following statements hold.