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Asymptotic nonlinear stability of traveling waves to a system of coupled Burgers equations[☆]

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A R T I C L E I N F O

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A B S T R A C T

This paper is concerned with the existence and asymptotic nonlinear stability of traveling wave solutions to a system of coupled Burgers equations which arises in a number of various physical contexts. We establish the existence of traveling fronts based on the phase plane analysis method. The asymptotic nonlinear stability of traveling wave solutions is proved by the method of energy estimates. In contrast to the previous related results, our results do not require any smallness assumption on the wave strengths or on the coefficients.

1. Introduction

In this paper, we investigate the existence and asymptotic nonlinear stability of traveling wave solutions to a system of coupled Burgers equations

$$\begin{cases} u_t + \left(\frac{1}{2}u^2 + \frac{1}{2}b^2\right)_x = \mu u_{xx}, \\ b_t + (ub)_x = \nu b_{xx}, \end{cases} \tag{1.1}$$

with the initial data

$$(u, b)(0, x) = (u_0, b_0)(x) \to \begin{cases} (u_-, b_-) & \text{as } x \to -\infty, \\ (u_+, b_+) & \text{as } x \to +\infty, \end{cases} \tag{1.2}$$

where μ and ν are two positive constants.

System (1.1) arises in a number of various physical contexts, for example, it is a simple model system coming from the theory of one-dimensional magnetohydrodynamic (MHD) turbulence. Since the full MHD equations are too complicated to investigate the small scale structure of the MHD turbulence, it is necessary to present simple model systems which contain essential features of the MHD turbulence. The system (1.1) is the simplest possible set of equations which allow "Alfvenization", i.e., the interchange of magnetic and fluid energies. It can be derived from the full MHD equations when the plasma density length scales are much longer than those of the magnetic field, resulting, to leading order, in a constant density; see [1] for details. Eqs. (1.1) may also model the opposite limit of a fluid-dominated (i.e., unmagnetized) system [2]. For broader applicability of (1.1), we refer the reader to Refs. [3–5] etc.

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In the limit $b \to 0$, system (1.1) reduces to nothing but the viscous Burgers equation, which was extensively studied by various authors (see, e.g., [6–8]). Note that the equations of the Elsässer variables $e^{\pm} = u \pm b$ can be written as

$$\frac{\partial e^{\pm}}{\partial t} + \frac{\partial}{\partial x}\frac{(e^{\pm})^2}{2} = \frac{\mu+\nu}{2}\frac{\partial^2 e^{\pm}}{\partial x^2} + \frac{\mu-\nu}{2}\frac{\partial^2 e^{\mp}}{\partial x^2}, \tag{1.3}$$

which mean, if $\mu = \nu$, that e^+ and e^- do not interact with each other in (1.3), and system (1.3) reduces to two independent viscous Burgers equations. The results in this paper include the more general case $\mu \neq \nu$ which is not integrable, suggesting a more complex and interesting interplay between the two fields u and b.

One of the purposes of the present paper is to prove rigorously the existence of traveling wave solutions to (1.1) with (1.2). We shall show this by using the method of phase plane analysis. The other purpose is to establish the asymptotic nonlinear stability of traveling wave solutions. The nonlinear stability problem of traveling wave solutions for hyperbolic systems has always attracted lots of interest and attention because of its physical importance and mathematical challenge. The study of stability of nonlinear waves to the Cauchy problems for scalar conservation laws was first proposed by Il'in and Oleinik [9]. In [10], Matsumura and Nishihara used the method of energy estimates to show that small-amplitude viscous shock profiles of conservation laws are asymptotically stable under zero-mass perturbations. Almost at the same time, also by means of the elementary energy method, a similar result but for weak shock profiles was obtained independently by Goodman [11]. Since then, this technique has been extensively developed and applied to a variety of models, for example, the Navier–Stokes equations and the more general system of viscous strictly hyperbolic conservation laws; see [12–18] etc., and some references therein. In this paper, we also use the method of energy estimates to establish the asymptotic nonlinear stability of traveling wave solutions. Our results, however, do not require any smallness hypothesis on the wave strengths. It is worthwhile to point out that the small wave strength is generally an assumption for the study of the nonlinear stability of traveling wave solutions.

The elementary energy method can also be used to deal with the stability problem of traveling wave solutions for the hyperbolic relaxation system. This problem was first considered by Liu [19] for a general model and then has been widely studied by many authors for the following so-called Jin–Xin relaxation system

$$\begin{cases} u_t + v_x = 0, \\ v_t + a u_x = \dfrac{1}{\varepsilon}(f(u) - v), \end{cases} \tag{1.4}$$

which was initially introduced by Jin and Xin [20] for numerical purposes. This model is considered as an ideal model problem to understand the general ones since it possesses the key features of a general relaxation system. The studies of the stability of traveling wave solutions for (1.4) and the related models are presented in [21–23] etc., and the references therein.

Recently, Li and Wang [24] have investigated a similar system to (1.1), that reads,

$$\begin{cases} u_t - (uv)_x = D u_{xx}, \\ v_t + (\varepsilon v^2 - u)_x = \varepsilon v_{xx}, \end{cases} \tag{1.5}$$

which is derived from the well-known Keller–Segel model describing the motion of the chemotaxis molds. Under the smallness hypothesis of ε, they have established the existence and nonlinear stability of traveling wave solutions to (1.5) without the smallness assumption on the wave strengths. Contrasting with the results of Li and Wang, our results are obtained without any smallness assumption on the coefficients.

Here we introduce some notations. Throughout this paper, C denotes the generic positive constant which changes from line to line. L^2 and $H^p (p \geq 0)$ denote the usual L^2-space and the Sobolev space with the norms $\|\cdot\|$ and $\|\cdot\|_p$, respectively.

The remainder of this paper is organized as follows. In Section 2, we state the main results of this paper including the existence and the nonlinear stability of traveling wave solutions to (1.1) with (1.2). We prove the existence of traveling wave solutions in Section 3. Section 4 is devoted to establishing the nonlinear stability of traveling wave solutions. Finally, conclusions are carried out in Section 5.

2. Preliminaries and main results

In this section, before stating our main results, we first provide some preliminary analysis, including the traveling wave solution with shock profile and its asymptotic stability. For simplicity, we restrict our attention to the following region

$$\mathbb{S} = \{(u, b) \mid u \geq 0,\ b \geq 0,\ u_{\pm} \geq 0,\ b_{\pm} \geq 0\}.$$

2.1. Existence of traveling waves

By direct calculation, one finds that the corresponding inviscid system of (1.1) has two real roots $\lambda_i(u, b) = u + (-1)^i b$, $i = 1, 2$ with respective right eigenvectors $r_i(u, b) = (1, (-1)^i)^{\top}$, $i = 1, 2$. Here we suppose that $b > 0$. The