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Fractional stochastic differential equations with applications to finance

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ABSTRACT

In this paper we use a definition of the fractional stochastic integral given by Carmona et al. (2003) in [19] and develop a simple approximation method to study quasi-linear stochastic differential equations by fractional Brownian motion. We also propose a stochastic process, namely fractional semimartingale, to model for the noise driving in some financial models.

1. Introduction

A fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a centered Gaussian process defined by

$$B_t^H = \int_0^t \bar{K}(t,s)dW_s, \tag{1.1}$$

where W is a standard Brownian motion and the kernel $\bar{K}(t,s)$, $t \geq s$, is given by

$$\bar{K}(t,s) = C_H\left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right)\int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}}du\right],$$

where $C_H = \sqrt{\frac{\pi H(2H-1)}{\Gamma(2-2H)\Gamma\left(H+\frac{1}{2}\right)^2 \sin\left(\pi\left(H-\frac{1}{2}\right)\right)}}$.

In [1] Mandelbrot and Van Ness have given a representation of B_t^H of the form:

$$B_t^H = \frac{1}{\Gamma(1+\alpha)}\left(Z_t + \int_0^t (t-s)^\alpha dW_s\right),$$

where $\alpha = H - \frac{1}{2}$, Z_t is a stochastic process of absolutely continuous trajectories, and $W_t^H := \int_0^t (t-s)^\alpha dW_s$ is called a Liouville fBm. A Liouville fBm shares many properties of a fBm except that it has non-stationary increments (for example, see [2]). Moreover, Comte and Renault in [3] have given an excellent application of Liouville fBm to finance. Because of these reasons and for simplicity we use W_t^H throughout this paper.

The main difficulty in studying fractional stochastic calculus is that we cannot apply stochastic calculus developed by Itô since fBm is neither a Markov process nor a semimartingale, except for $H = \frac{1}{2}$. Recently, there have been numerous attempts to define a stochastic integral with respect to fBm. The main approaches are the following ones (refer [4,5] for a detailed survey).

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- The pathwise approach was introduced by Lin [6]. Since the trajectories of the fBm are β-Hölder continuous, $\beta < H$ and by the work of Young in [7], the pathwise Riemann–Stieltjes integral exists for any integrand with sample paths γ-Hölder continuous with $\gamma + H > 1$. In [8] Nualart and Răşcanu have studied stochastic differential equations with respect to fBm with Hurst index $H > 1/2$.
- The regularization approach was introduced by Russo and Vallois in [9,10] and further developed by Cheridito and Nualart in [11]. This approach has been used by Nourdin in [12] to prove the existence and uniqueness of stochastic differential equations and an approximation scheme for all $H \in (0, 1)$.
- The Malliavin calculus approach was introduced by Decreusefond and Üstünel in [13] for fBm and extended to more general Gaussian processes by Alòs et al. in [14], Decreusefond in [15], etc. This approach leads to some different definitions for the fractional stochastic integrals such as the divergence integral, the Skorohod integral and the Stratonovich integral. By using the Skorohod integral some stochastic differential equations have been studied in [16,17] for the case of linear equations and in [18] for the case of quasi-linear equations with Hurst index $H < 1/2$.

It is known that the study of the stochastic differential equations (SDEs) depends on the definitions of the stochastic integrals involved. One of the definitions of the fractional stochastic integrals is given by Carmona et al. in [19]. This kind of fractional stochastic integral belongs to the third approach mentioned above and turns out to be equal to the divergence integral plus a complementary term (see Remark 18 in [4]). Thus it can be considered as a new definition of fractional stochastic integrals and naturally, the theory of SDEs needs studying independently.

In this paper we use Carmona et al.'s definition to study the SDEs driven by fractional Brownian motion. When the integrand is deterministic, our fractional stochastic integral coincides with the Wiener integral and SDEs of the form

$$dX_t = b(t, X_t)dt + \sigma(t)dW_t^H, \qquad X_0 = x_0, \tag{1.2}$$

have been studied by Mishura (Section 3.5 in [5]). As a new contribution to (1.2), we will point out a way to find explicitly its solution.

More generally, our work deals with the following form of SDEs:

$$dX_t = b(t, X_t)dt + \sigma(t)X_t dW_t^H, \qquad X_0 = x_0. \tag{1.3}$$

In order to prove the existence and uniqueness of the solution of Eq. (1.3) we make the following standard assumptions on coefficients. The volatility $\sigma : [0, T] \longrightarrow \mathbb{R}$ is a deterministic function on $[0, T]$, bounded by a constant M and the drift coefficient $b : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function in all their arguments and satisfies the following conditions, for a positive constant L_0:

(C_1) $b(t, x)$ is a continuously differentiable function in x and

$$|b(t, x) - b(t, y)| \le L_0|x - y| \tag{1.4}$$

for all $x, y \in \mathbb{R}$, $t \in [0, T]$;

(C_2) linear growth

$$|b(t, x)| \le L_0(1 + |x|), \quad \forall x \in \mathbb{R},\ \forall t \in [0, T]. \tag{1.5}$$

Since Eq. (1.3) is an anticipate SDE, similar to the Brownian case, the traditional methods cannot be applied. A method of approximation equations has been introduced recently by Thao (see [20] and the references therein) to solve simple linear SDEs and then used by N.T. Dung to solve more complicated SDEs such as the fractional SDEs with polynomial drift [21] and the fractional geometric mean-reversion equations [22]. We continue to develop this method in current work with the main idea being that the existence and uniqueness of the solution for Eq. (1.3) can be proved via an "approximation" equation, which is driven by semimartingales, and that the limit in $L^2(\Omega)$ of approximation solution will be the solution of (1.3). Thus, advantages of this method are that we can still use classical Itô calculus and do not need any other fractional stochastic calculus. Based on our obtained approximation results, we propose a stochastic process, namely fractional semimartingale, to model for noise driving in some financial models.

This paper is organized as follows. In Section 2, we recall the definition of fractional stochastic integral given in [19] and some moment inequalities for fractional stochastic integral of deterministic integrands. Section 3 contains the main result of this paper which proved the existence and uniqueness of the solution of Eq. (1.3). In Section 4, the European option pricing formula in the fractional semimartingale Black–Scholes model is found and the optimal portfolio in a stochastic drift model is investigated.

2. Preliminaries

Let us recall some elements of stochastic calculus of variations.

For $h \in L^2([0, T], \mathbb{R})$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t)dW_t.$$