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A variational approach via bipotentials for unilateral contact problems

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ABSTRACT

We consider a unilateral contact model for nonlinearly elastic materials, under the small deformation hypothesis, for static processes. The contact is modeled with Signorini's condition with zero gap and the friction is neglected on the potential contact zone. The behavior of the material is modeled by a subdifferential inclusion, the constitutive map being proper, convex, and lower semicontinuous. After describing the model, we give a weak formulation using a bipotential which depends on the constitutive map and its Fenchel conjugate. We arrive to a system of two variational inequalities whose unknown is the pair consisting of the displacement field and the Cauchy stress field. We look for the unknown into a Cartesian product of two nonempty, convex, closed, unbounded subsets of two Hilbert spaces. We prove the existence and the uniqueness of the weak solution based on minimization arguments for appropriate functionals associated with the variational system. How the proposed variational approach is related to previous variational approaches, is discussed too.

1. Introduction

We consider a 3D elastostatic frictionless unilateral contact model, under the small deformation hypothesis, for nonlinearly elastic materials. The mechanical model is described mathematically by a boundary value problem consisting of a system of partial differential equations associated with a displacement boundary condition, a traction boundary condition and a contact condition. The contact is modeled by Signorini's contact condition with zero gap neglecting the friction on the potential contact zone. The behavior of the material is expressed by a constitutive law which involves a nonlinear elastic operator, possibly multi-valued; more precisely, the stress belongs to the subdifferential of a proper, convex, lower semicontinuous functional.

The analyses of the elastostatic frictionless unilateral contact model for single-valued elastic operators can be found in the literature into the frame of variational inequalities of the first kind. We recall that the weak formulation in displacements, called the *primal variational formulation*, is a nonhomogeneous variational inequality of the first kind and has a unique solution; the weak formulation in terms of stress, called the *dual variational formulation*, is a homogeneous variational inequality and has a unique solution, too (details can be found e.g. in [1]). For multi-valued elastic operators the model was studied using a variational formulation in displacements and a minimization technique; see e.g. [2].

In the present paper we focus on the weak solvability of the model giving a new weak formulation which consists of a system of two variational inequalities. The unknown of this system is the pair $(\boldsymbol{u}, \boldsymbol{\sigma}) \in K \times \Lambda$ where $\boldsymbol{u}$ is the displacement field, $\boldsymbol{\sigma}$ is the stress field, and K, Λ are two nonempty, convex, closed, unbounded subsets of two Hilbert spaces. The key of the proposed approach is a *bipotential function* which depends on the constitutive map and its Fenchel conjugate. The construction of several bipotential functions appears in connection with Coulomb's friction law [3] and Cam-Clay models in soil mechanics [4,5], cyclic plasticity [6,7] and viscoplasticity of metals with nonlinear kinematical hardening rule [8],

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Lemaitre's damage law [9], the coaxial laws [10,11], and the elastic–plastic bipotential of soils [12]. Other interesting results involving the bipotential theory can be found in the overview paper [13] or in the recent paper [14], as well as in the paper [15].

The present paper extends and improves the results in [15]; there the authors illustrate the applicability of a bipotential function to the solution of a displacement–traction model in elastostatics. The model in the present paper is more complex; unlike the model in [15], it takes into account a possible contact (on a part of the boundary) between the deformable body and a rigid obstacle. This particularity of the model requires a new functional setting. In this new functional setting we prove the existence and the uniqueness of the weak solution of the unilateral contact model. The new variational approach, which is a global weak formulation of the model, allows to compute simultaneously the displacement field and the Cauchy stress tensor based on minimization arguments for appropriate functionals associated with the variational system. The proposed variational approach is suitable to apply numerical methods of multigrid type in order to approximate the weak solution. Also, it is worth to be mentioned that the proposed variational approach covers the previous variational approaches. As we shall see in Section 3 of the present paper, for a class of nonlinear materials, the weak solution via bipotentials coincides with the pair consisting of the unique solution of the primal variational formulation and the unique solution of the dual variational formulation.

To follow the present exposure easily, we mention here the main tools we use and some useful references.

Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ be a Hilbert space. A *bipotential* is a function $B : X \times X \to (-\infty, \infty]$ with the following three properties:

(i) B is convex and lower semicontinuous in each argument;
(ii) for each $x, y \in X$, we have $B(x, y) \geq (x, y)_X$;
(iii) for each $x, y \in X$, $y \in \partial B(\cdot, y)(x) \Leftrightarrow x \in \partial B(x, \cdot)(y) \Leftrightarrow B(x, y) = (x, y)_X$.

The *Fenchel conjugate* of a functional $\phi : X \to (-\infty, \infty]$ is the functional

$$\phi^* : X \to (-\infty, \infty], \quad \phi^*(x^*) = \sup_{x\in X} \left\{ \left(x^*, x\right) - \phi(x) \right\}.$$

Theorem 1. *Let $\phi : X \to (-\infty, \infty]$ be a proper, convex, lower semicontinuous functional. Then:*

(i) *for each $x, y \in X$, we have $\phi(x) + \phi^*(y) \geq (x, y)_X$;*
(ii) *for each $x,\ y \in X$, $y \in \partial\phi(x) \Leftrightarrow x \in \partial\phi^*(y) \Leftrightarrow \phi(x) + \phi^*(y) = (x, y)_X$.*

Also, we shall need the following theorem.

Theorem 2. *Let $(X, \|\cdot\|_X)$ be a reflexive Banach space and let $K \subset X$ be a nonempty, convex, closed, unbounded subset of X. Suppose $\varphi : K \to \mathbb{R}$ is coercive, convex and lower semicontinuous. Then φ is bounded from below on K and attains its infimum in K. If φ is strictly convex then φ has a unique minimizer.*

Minimization results can be found in many books, see e.g., [16,17]; for additional elements of convex analysis the reader can consult also, e.g. [18–21]. We recall that $\varphi : K \to \mathbb{R}$ is coercive if, for all $u \in K$ we have $\varphi(u) \to \infty$ as $\|u\|_X \to \infty$.

To increase the clarity of the exposure the paper was written in a self-contained manner. However, we mention here that, in addition to elements of convex analysis and calculus of variations, the present paper requires a background of mechanics of solid and mathematical theory of contact problems which can be covered from, e.g., [22,23,1,24].

We end this introductory part by mentioning the structure of the rest of the paper. In Section 2 we state the mechanical model and we prove the existence and the uniqueness of the weak solution. In Section 3 we discuss how the proposed variational approach is related to the previous variational approaches: the primal variational formulation and the dual variational formulation.

2. The model and its weak solvability via bipotentials

We consider a body that occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary, partitioned in three measurable parts, Γ_1, Γ_2 and Γ_3, such that the Lebesgue measure of Γ_1 is positive. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body can be in contact with a rigid foundation. According to this physical setting we formulate the following boundary value problem.

Problem 1. Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, such that

$$\mathrm{Div}\,\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} \quad \text{in } \Omega, \tag{1}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{3}$$

$$\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) \quad \text{on } \Gamma_2, \tag{4}$$

$$\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = \boldsymbol{0}, \qquad u_\nu(\boldsymbol{x}) \leq 0, \qquad \sigma_\nu(\boldsymbol{x}) \leq 0, \qquad \sigma_\nu(\boldsymbol{x}) u_\nu(\boldsymbol{x}) = 0 \quad \text{on } \Gamma_3. \tag{5}$$