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Note

Lie symmetries of systems of second-order linear ordinary differential equations with constant coefficients

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ABSTRACT

Lie symmetries of systems of second-order linear ordinary differential equations with constant coefficients are exhaustively described over both the complex and real fields. The exact lower and upper bounds for the dimensions of the maximal Lie invariance algebras possessed by such systems are obtained using an effective algebraic approach.

1. Introduction

The problem on possible dimensions of the maximal Lie invariance algebras of differential equations from a fixed class has a long history. Already S. Lie obtained exhaustive results concerning maximal dimensions of such algebras for ordinary differential equations (ODEs) of any fixed order [1, S. 294–301]. Namely, he first proved that any first-order ODE possesses an infinite-dimensional Lie invariance algebra, the dimension of the maximal Lie invariance algebra of any second-order ODE (resp. any mth order ODE for $m \geqslant 3$) is at most eight (resp. not greater than $m+4$), and these bounds are exact. Later these results were repeatedly reinvented; see e.g. [2].

Analogous results for systems of ODEs are much less known. We discuss some of them, which are relevant to the subject of the present paper. Thus, according to the remarkable lecture notes by Markus [3, pp. 68–69, Theorem 44], any system of second-order ODEs

$$\ddot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}), \tag{1}$$

where $\boldsymbol{x}(t) = (x^1(t), \ldots, x^n(t))^{\mathrm{T}}$, $\dot{\boldsymbol{x}} = d\boldsymbol{x}/dt$, $\ddot{\boldsymbol{x}} = d\dot{\boldsymbol{x}}/dt$, possesses the maximal Lie invariance algebra of dimensions not greater than $(n+2)^2 - 1$. This result was later reproved in [2, Sections 4 and 5]. It was also shown therein that the maximal dimension $(n+2)^2 - 1 = n^2 + 4n + 3$ is reached for systems reduced by point transformations to the simplest system

$$\ddot{\boldsymbol{x}} = \boldsymbol{0}. \tag{2}$$

The maximal Lie invariance algebra $\mathfrak{g}^0$ of the system (2) is generated by the vector fields

$$\partial_t, \qquad \partial_{x^a}, \qquad t\partial_t, \qquad x^a\partial_t, \qquad t\partial_{x^a}, \qquad x^a\partial_{x^b}, \qquad tx^a\partial_t + x^a x^c\partial_{x^c}, \qquad t^2\partial_t + tx^c\partial_{x^c},$$

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and is isomorphic to the Lie algebra $\mathrm{sl}(n+2,\mathbb{C})$ (resp. $\mathrm{sl}(n+2,\mathbb{R})$) for the complex (resp. real) case; see e.g. [4]. Here and in what follows the indices a, b, c run from 1 to n, i.e. $a, b, c = 1, \ldots, n$, and we use the summation convention for repeated indices. It can be checked that the system (2) is invariant with respect to the general projective group of $\mathbb{C}^{n+1}$ (resp. $\mathbb{R}^{n+1}$) consisting of the point transformations [5, S. 554]

$$\tilde{x}^i = \frac{\alpha_{i0}x^0 + \cdots + \alpha_{in}x^n + \alpha_{i,n+1}}{\alpha_{n+1,0}x^0 + \cdots + \alpha_{n+1,n}x^n + \alpha_{n+1,n+1}}, \quad i = 0, \ldots, n,$$

where $\alpha_{00}, \alpha_{01}, \ldots, \alpha_{n+1,n+1}$ are homogeneous group parameters and $x^0 = t$, and $\mathfrak{g}^0$ is the Lie algebra associated with this group. In fact, the number of essential group parameters is $(n+2)^2 - 1$ as, supposing a homogeneous group parameter nonzero, we can set a nonzero parameter to be equal to 1, simultaneously dividing of the numerator and the denominator in the expression for each $\tilde{x}^i$ by this parameter and then assigning parameter ratios as new parameters.

Fels proved [6,7] that up to point equivalence the system (2) is a unique system of the form (1) which admits an (n^2+4n+3)-dimensional Lie invariance algebra, which was earlier known only for linear systems [4]. Recently linearization criteria for systems from the class (1) have been independently investigated in [8–10]. See also the references therein. The maximal dimension of the maximal Lie invariance algebras for normal systems of mth order ODEs was estimated in [2,11,12] for an arbitrary $m \geqslant 3$, and it is known for the case $m = 3$ that up to point equivalence the system $\dddot{\boldsymbol{x}} = 0$ is a unique system for which the dimension of the maximal Lie invariance algebra reaches the maximal value $n^2 + 3n + 3$ for such systems [6].

In a series of recent papers the study of Lie symmetries of systems of n ($n \geqslant 2$) linear second-order ODEs with constant coefficients was recovered. Namely, the cases $n = 2$ and $n = 3$ were considered in [13]. In [14] Campoamor-Stursberg corrected results of [13] (see also comments concerning [13] in [15]), and studied the case $n = 4$ as well as systems associated with diagonal matrices without restrictions on n. Certain results on the dimensions of the maximal Lie invariance algebras of such systems in the case of arbitrary n and matrices of the general Jordan form were obtained in [16].

The study of symmetry properties of systems from the class (1) is required by numerous applications in mechanics, gravity, etc. Unfortunately, there are no general results on Lie symmetries of these systems. This is why even linear systems with constant coefficients are good objects for a preliminary investigation in spite of the well-known simple algorithm for the construction of their general solutions. Group classification of linear systems with constant coefficients gives examples what Lie algebras of vector fields are admitted by systems from the class (1) as their maximal Lie invariance algebras and what dimensions of these algebras are possible. Note that the above knowledge is important for the problem on linearization of systems from the class (1).

The purpose of the present paper, which is inspired by the papers [13–16], is to exhaustively describe Lie symmetries of systems of second-order linear ordinary differential equations with constant coefficients over both the complex and real fields. We essentially enhance and generalize the results of [13–16] using a simple but effective algebraic approach. In particular, we explicitly describe the maximal Lie invariance algebras possessed by systems under consideration with no restrictions on the number of equations and the form of system coefficients and derive the exact lower and upper bounds for the dimensions of these algebras.

2. Main result

Following [13–16], we consider systems of linear second-order ODEs of the normal form

$$\ddot{\boldsymbol{x}} = A\dot{\boldsymbol{x}} + B\boldsymbol{x} + \boldsymbol{C}(t) \tag{3}$$

over the complex field. Here $\boldsymbol{C}(t)$ is a smooth n-component vector-function of t, and A and B are constant complex-valued matrices of dimension $n \times n$, $n \geqslant 2$. Note that the choice of the underlying field ($\mathbb{C}$ or $\mathbb{R}$) is not principal. We choose the complex field in order to make the presentation clearer.

It is commonly known (see, e.g., [4]) that the change of dependent variables $\boldsymbol{x} = \exp(\frac{1}{2}At)\boldsymbol{y} + \boldsymbol{x}_\mathrm{p}(t)$, where $\boldsymbol{x}_\mathrm{p}(t)$ is a particular solution of (3), maps (3) into the system

$$\ddot{\boldsymbol{y}} = D\boldsymbol{y}, \quad \text{where } D = B - A^2.$$

By J we denote the Jordan normal form of the matrix D. Then there exists a nondegenerate matrix P such that $D = P^{-1}JP$, and the point transformation $\boldsymbol{y} = P\boldsymbol{z}$ reduces the last system to the form $\ddot{\boldsymbol{z}} = J\boldsymbol{z}$. As a result, for the study of symmetry properties of normal systems of linear second-order ODEs with constant coefficients it suffices to consider only the systems of the form

$$\ddot{\boldsymbol{x}} = J\boldsymbol{x}, \tag{4}$$

where J is a Jordan matrix,

$$J = \bigoplus_{l=1}^{s} J_{\lambda_l}^{k_l}, \qquad k_1 + \cdots + k_s = n. \tag{5}$$