



On $\mathcal{L}_\infty$ -envelopes of Banach spaces

Jesús M.F. Castillo^{a,*}, Jesús Suárez^b

^a Departamento de Matemáticas, Facultad de Ciencias, Univ. de Extremadura, Avda. de Elvas s/n, 06071 Badajoz, Spain

^b Escuela Politécnica, Universidad de Extremadura, Avda. de la Universidad s/n, 10071 Cáceres, Spain

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ABSTRACT

Let $\mathcal{A}$ denote any of the following classes of $\mathcal{L}_\infty$ -spaces: $C(K)$ -spaces, Lindenstrauss spaces, λ -separably injective spaces, universally λ -separably injective spaces, λ -Lindenstrauss–Pełczyński spaces or $\mathcal{L}_{\infty,\lambda}$ -spaces. We show that every Banach space X can be isometrically embedded into a space $\mathcal{A}(X) \in \mathcal{A}$ so that every operator $X \rightarrow A$ with $A \in \mathcal{A}$ can be extended to an operator $\mathcal{A}(X) \rightarrow A$ with the same norm.

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1. Introduction and preliminaries

As is well-known, every Banach space X can be naturally embedded into the space of continuous functions $C(B_{X^*})$ in such a way that the embedding $\delta_X : X \rightarrow C(B_{X^*})$ enjoys the following universal property:

Every $C(K)$ – valued operator $\tau : X \rightarrow C(K)$ can be extended through δ_X to an operator

$T : C(B_{X^*}) \rightarrow C(K)$ so that $\|T\| = \|\tau\|$.

See [1], and also [2] for a historical account and a categorical presentation of this result. Does there exist a similar universal embedding for other classes of Banach spaces? It is particularly interesting to consider the situation replacing $C(K)$ -spaces by their “local” versions, the $\mathcal{L}_\infty$ spaces. Recall that a Banach space X is said to be an $\mathcal{L}_{\infty,\lambda}$ -space, $\lambda \geq 1$, if every finite dimensional subspace F of X is contained in another finite dimensional subspace G of X whose Banach–Mazur distance to the corresponding $\ell_\infty^{\dim G}$ is at most λ . A space X is said to be an $\mathcal{L}_\infty$ -space if it is an $\mathcal{L}_{\infty,\lambda}$ -space for some $\lambda \geq 1$. The following subclasses of $\mathcal{L}_\infty$ -spaces have appeared in the literature (the missing definitions can be found in the next paragraph): $\mathcal{L}_{\infty,\lambda}$ -spaces, λ -Lindenstrauss–Pełczyński spaces, Lindenstrauss spaces, $C(K)$ -spaces, λ -separably injective spaces, universally λ -separably injective spaces and λ -injective spaces. A Banach space is said to be a Lindenstrauss space if it is an $\mathcal{L}_{\infty,1+\varepsilon}$ -space for all $\varepsilon > 0$. The space X is said to be injective if every operator $\tau : Y \rightarrow X$ can be extended to an operator $T : W \rightarrow X$ for any bigger superspace W of Y . The space X is said to be separably injective (resp. universally separably injective) [3] when the condition above is satisfied for separable W (resp. separable) Y . It is not hard to see [3] that for separable injectivity it is enough to take $W = \ell_1$; and $W = \ell_\infty$ for universal separable injectivity. The space X is said to be a Lindenstrauss–Pełczyński space [4,5] when the previous condition holds for $W = c_0$. When an extension T verifying $\|T\| \leq \lambda \|\tau\|$ exists then the classes will be referred to, respectively, as λ -injective, λ -separably injective, universally λ -separably injective, and λ -Lindenstrauss–Pełczyński spaces.

Definition 1.1. Let $\mathcal{A}$ be a class of Banach spaces. An $\mathcal{A}$ -envelope of a Banach space X is a couple $(\mathcal{A}(X), \delta)$ formed by a Banach space $\mathcal{A}(X) \in \mathcal{A}$ and an isometric embedding $\delta : X \rightarrow \mathcal{A}(X)$ with the property that for every $A \in \mathcal{A}$ every operator $\tau : X \rightarrow A$ can be extended through δ – i.e., $T\delta = \tau$ – to an operator $T : \mathcal{A}(X) \rightarrow A$ such that $\|T\| = \|\tau\|$.

* Corresponding author.

E-mail addresses: castillo@unex.es (J.M.F. Castillo), jesus@unex.es (J. Suárez).

The problem addressed and affirmatively solved in this paper is the following.

Problem. Let $\mathcal{A}$ denote any of the following classes of $\mathcal{L}_\infty$ -spaces: $C(K)$ -spaces, Lindenstrauss spaces, λ -separably injective spaces, universally λ -separably injective spaces, λ -Lindenstrauss–Pełczyński spaces, $\mathcal{L}_{\infty,\lambda}$ -spaces. Does there exist an $\mathcal{A}$ -envelope of any given Banach space X ?

To this end, we will develop a rather flexible device to construct such envelopes which unifies several constructions presented or outlined in the literature (cf., [6–11]). The final Remark in [7] can be understood as a hint that such unified approach was possible.

The results in this paper solve the problems posed by the first author in his lecture during the Workshop in Topology and Banach spaces, celebrated in Łódź, Poland, 21–25 July 2010, and organized by P. Koszmider. It is a pleasure to thank Prof. Koszmider for the organization of such event and the invitation to participate. Moreover, the first author lectured about the results in this paper at the Banach space workshop celebrated at Banff International Research Station for Mathematical Innovation and Discovery (BIRS) in Banff, Alberta, Canada; from 4 to 9 March 2012. It is a pleasure to thank the organizers R. Anisca, S. Dilworth, E. Odell and B. Sari for the invitation to participate. Special thanks must go to the scientific director of BIRS, Prof. N. Ghoussoub and to BIRS Station Manager Brenda Williams for their exceptionally warm hospitality.

The basic for our purposes is the push-out construction, which naturally appears when one considers a couple – or a family – of operators defined on the same space; in particular when considering the extension of an operator through an embedding. Let us explain why. Given operators $\alpha : Y \rightarrow A$ and $\beta : Y \rightarrow B$, the associated push-out diagram is

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \beta' \\ B & \xrightarrow{\alpha'} & \text{PO}. \end{array} \quad (1)$$

Here, the push-out space $\text{PO} = \text{PO}(\alpha, \beta)$ is the quotient of the direct sum $A \oplus_1 B$, the product space endowed with the sum norm, by the closure of the subspace $\Delta = \{(\alpha y, -\beta y) : y \in Y\}$. The map α' is given by the inclusion of B into $A \oplus_1 B$ followed by the natural quotient map $A \oplus_1 B \rightarrow (A \oplus_1 B)/\Delta$, so that $\alpha'(b) = (0, b) + \overline{\Delta}$ and, analogously, $\beta'(a) = (a, 0) + \overline{\Delta}$. The diagram (1) is commutative: $\beta'\alpha = \alpha'\beta$. Moreover, it is ‘minimal’ in the sense of having the following universal property: if $\beta'' : A \rightarrow C$ and $\alpha'' : B \rightarrow C$ are operators such that $\beta''\alpha = \alpha''\beta$, then there is a unique operator $\gamma : \text{PO} \rightarrow C$ such that $\alpha'' = \gamma\alpha'$ and $\beta'' = \gamma\beta'$. Clearly, $\gamma((a, b) + \overline{\Delta}) = \beta''(a) + \alpha''(b)$ and one has $\|\gamma\| \leq \max\{\|\alpha''\|, \|\beta''\|\}$. Regarding the behavior of the maps in diagram (1), one has [6] the following lemma.

Lemma 1.2. (1) $\max\{\|\alpha'\|, \|\beta'\|\} \leq 1$.

(2) If α is an isomorphic embedding, then Δ is closed.

(3) If α is an isometric embedding and $\|\beta\| \leq 1$ then α' is an isometric embedding.

(4) If α is an isomorphic embedding then α' is an isomorphic embedding.

(5) If $\|\beta\| \leq 1$ and α is an isomorphism then α' is an isomorphism and

$$\|(\alpha')^{-1}\| \leq \max\{1, \|\alpha^{-1}\|\}.$$

Returning to the using of the push-out construction, applying the previous situation when α is an isomorphic embedding and β an operator, we get that β' is an “extension” of β , except for the fact that one needs to increase the target space from B to PO . Nevertheless this increasing is as small and controlled as possible: indeed, it is not difficult to show [12] that PO/B and A/Y are isomorphic.

The following result shows that it is enough to obtain envelopes of separable spaces. Let us recall that given an ordinal μ , a union $\bigcup_{\alpha < \mu} X_\alpha$ is said to be continuous if for every limit ordinal $\beta < \mu$, $X_\beta = \overline{\bigcup_{\alpha < \beta} X_\alpha}$.

Lemma 1.3. If separable spaces admit separable $\mathcal{A}$ -envelopes then all Banach spaces admit $\mathcal{A}$ -envelopes.

Proof. We proceed by transfinite iteration on the density character of the space. For the case $\text{dens}(X) = \aleph_0$, i.e., X separable, the existence of a separable $\mathcal{A}$ -envelope $\mathcal{A}(X)$ is guaranteed by the hypothesis. Let κ be an ordinal, and assume that $\mathcal{A}$ -envelopes of spaces X with $\text{dens}(X) = \aleph_\kappa$ do exist. If $\text{dens}(X) = \aleph_{\kappa+1}$, let $\omega_{\kappa+1}$ be the first ordinal with cardinal $|\omega_{\kappa+1}| = \aleph_{\kappa+1}$. Then X can be represented as an increasing continuous union

$$X = \bigcup_{\alpha < \omega_{\kappa+1}} X_\alpha$$

of spaces with $\text{dens}(X_\alpha) \leq \aleph_\kappa$. The passage from α to $\alpha + 1$ is described in the diagram

$$\begin{array}{ccccc} X_\alpha & \longrightarrow & X_{\alpha+1} & & \\ \delta_\alpha \downarrow & & \downarrow \delta_{\alpha'} & & \\ \mathcal{A}(X_\alpha) & \longrightarrow & \text{PO}_\alpha & \xrightarrow[\Delta_\alpha]{} & \mathcal{A}(\text{PO}_\alpha). \end{array}$$

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