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Homogenization of a stochastic nonlinear reaction–diffusion equation with a large reaction term: The almost periodic framework

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ABSTRACT

Homogenization of a stochastic nonlinear reaction–diffusion equation with a large nonlinear term is considered. Under a general Besicovitch almost periodicity assumption on the coefficients of the equation we prove that the sequence of solutions of the said problem converges in probability towards the solution of a rather different type of equation, namely, the stochastic nonlinear convection–diffusion equation which we explicitly derive in terms of appropriate functionals. We study some particular cases such as the periodic framework, and many others. This is achieved under a suitable generalized concept of Σ-convergence for stochastic processes.

1. Introduction

Homogenization theory is an important branch of asymptotic analysis. Since the pioneering work of Bensoussan et al. [1] it has grown very significantly, giving rise to several sub-branches such as the *deterministic* homogenization theory and the *random* homogenization theory. Each of these sub-branches has been developed and deepened. Regarding the deterministic homogenization theory, from the classical periodic theory [1] to the recent general deterministic ergodic theory [2–5], many results have been reported and continue to be published. We refer to some of these results [6,2–5] relating to the deterministic homogenization of deterministic partial differential equations in the periodic framework and in the deterministic ergodic framework in general.

The random homogenization theory is divided into two major subgroups: the homogenization of differential operators with random coefficients, and the homogenization of stochastic partial differential equations. As far as the first subgroup is concerned, many results have been obtained to date; we refer e.g. to [7–17].

In contrast with either the deterministic homogenization theory or the homogenization of partial differential operators with random coefficients, very few results are available in the setting of the homogenization of stochastic partial differential equations (SPDEs). We cite for example the works [18–22] which consider the homogenization problems related to SPDEs with periodic coefficients (only!). Homogenization of SPDEs with non oscillating coefficients was considered in [23] in domains with non periodically distributed holes. The approach in [23] is different and is the stochastic version of Marchenko–Khruslov–Skrypnik's theory developed in [24,25]. It should be noted that unfortunately for SPDEs with oscillating coefficients no results are available beyond the periodic setting.

Given the significance of SPDEs in modeling of physical phenomena, in addition to simple random periodically perturbed phenomena, it is important to think of a theory generalizing that of the homogenization of SPDEs with periodic coefficients. This is one of the objectives of this work.

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More precisely, we discuss the homogenization problem for the following nonlinear SPDE

$$\begin{cases} du_\varepsilon = \left(\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)Du_\varepsilon\right) + \frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)\right)dt + M\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)dW \quad \text{in } Q_T \\ u_\varepsilon = 0 \quad \text{on } \partial Q \times (0,T) \\ u_\varepsilon(x,0) = u^0(x) \quad \text{in } Q \end{cases} \tag{1.1}$$

in the almost periodic environment, where $Q_T = Q \times (0,T)$, Q being a Lipschitz domain in $\mathbb{R}^N$ with smooth boundary ∂Q, T is a positive real number and W is a m-dimensional standard Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The choice of the above problem lies in its application in engineering (see for example [26–28] in the deterministic setting, and [29] in the stochastic framework, for more details). In fact, as in [26], the unknown u_ε may be viewed as the concentration of some chemical species diffusing in a porous medium of constant porosity, with diffusivity $a(y,\tau)$ and reacting with background medium through the nonlinear term $g(y,\tau,u)$ under the influence of a random external source $M(y,\tau,u)dW$ (M is a $\mathbb{R}^m$-valued function and throughout $M(y,\tau,u)dW$ will denote the scalar product of M and dW in $\mathbb{R}^m$). The motivation of this choice is several fold. Firstly, we start from a SPDE of reaction–diffusion type, and we end up, after the passage to the limit, with a SPDE of a convection–diffusion type; this is because of the large reaction's term $\frac{1}{\varepsilon}g(x/\varepsilon, t/\varepsilon^2, u_\varepsilon)$ which satisfies some kind of centering condition; see Section 4 for details. Secondly, the order of the microscopic time scale here is twice that of the microscopic spatial scale. This leads after the passage to the limit, to a rather complicated so-called cell problem, which is besides, a deterministic parabolic type equation, the random variable behaves in the latter equation just like a parameter. Such a problem is difficult to deal with as, in our situation, it involves a microscopic time derivative derived from the semigroup theory, which is not easy to handle. Thirdly, in order to solve the homogenization problem under consideration, we introduce a suitable type of convergence which takes into account both deterministic and random behavior of the data of the original problem. This method is formally justified by the theory of Wiener chaos polynomials [30,31]. In fact, following [30] (see also [31]), any sequence of stochastic processes $u^\varepsilon(x,t,\omega) \in L^2(Q\times(0,T)\times\Omega)$ expresses as follows:

$$u^\varepsilon(x,t,\omega) = \sum_{j=1}^{\infty} u_j^\varepsilon(x,t)\Phi_j(\omega)$$

where the functions Φ_j are the generalized Hermite polynomials, known as the Wiener-chaos polynomials. The above decomposition clearly motivates the definition of the concept of convergence used in this work; see Section 3 for further details. Finally, the periodicity assumption on the coefficients is here replaced by the almost periodicity assumption. Accordingly, it is the first time that an SPDE is homogenized beyond the classical period framework, and our result is thus, new. It is also important to note that in the deterministic framework, i.e. when $M = 0$ in (1.1), the equivalent problem obtained has just been solved by Allaire and Piatnitski [26] under the periodicity assumption on the coefficients, but with a weight function on the derivative with respect to time. Our results therefore generalize to the almost periodic setting, those obtained by Allaire and Piatnitski in [26].

The layout of the paper is as follows. In Section 2 we recall some useful facts about almost periodicity that will be used in the next sections. Section 3 deals with the concept of Σ-convergence for stochastic processes. In Section 4, we state the problem to be studied. We proved there a tightness result that will be used in the next section. We state and prove homogenization results in Section 5. In particular we give in that section the explicit form of the homogenization equation. Finally, in Section 6, we give some applications of the result obtained in the previous section.

Throughout Section 2, vector spaces are assumed to be complex vector spaces, and scalar functions are assumed to take complex values. We shall always assume that the numerical space $\mathbb{R}^m$ (integer $m \geq 1$) and its open sets are each equipped with the Lebesgue measure $dx = dx_1 \dots dx_m$.

2. Spaces of almost periodic functions

The concept of almost periodic functions is well known in the literature. We present in this section some basic facts about it, which will be used throughout the paper. For a general presentation and an efficient treatment of this concept, we refer to [32–34].

Let $\mathcal{B}(\mathbb{R}^N)$ denote the Banach algebra of bounded continuous complex-valued functions on $\mathbb{R}^N$ endowed with the sup norm topology.

A function $u \in \mathcal{B}(\mathbb{R}^N)$ is called a almost periodic function if the set of all its translates $\{u(\cdot + a)\}_{a\in\mathbb{R}^N}$ is precompact in $\mathcal{B}(\mathbb{R}^N)$. The set of all such functions forms a closed subalgebra of $\mathcal{B}(\mathbb{R}^N)$, which we denote by $AP(\mathbb{R}^N)$. From the above definition, it is an easy matter to see that every element of $AP(\mathbb{R}^N)$ is uniformly continuous. It is classically known that the algebra $AP(\mathbb{R}^N)$ enjoys the following properties:

(i) $\overline{u} \in AP(\mathbb{R}^N)$ whenever $u \in AP(\mathbb{R}^N)$, where $\overline{u}$ stands for the complex conjugate of u;
(ii) $u(\cdot + a) \in AP(\mathbb{R}^N)$ for any $u \in AP(\mathbb{R}^N)$ and each $a \in \mathbb{R}^N$.