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Three non-zero solutions for a nonlinear eigenvalue problem

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ABSTRACT

In the present paper we prove a novel multiplicity result for a model quasilinear Dirichlet problem (P_λ) depending on a positive parameter λ. By a variational method, we prove that for every $\lambda > 1$ problem (P_λ) has at least two non-zero solutions, while there exists $\hat{\lambda} > 1$ such that problem $(P_{\hat{\lambda}})$ has at least three non-zero solutions.

1. Introduction

In the present paper we deal with the problem of multiplicity results for the following quasilinear equation coupled with the Dirichlet boundary condition

$$\begin{cases} -\Delta_p u = \lambda \alpha(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a bounded open connected set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p > n$, Δ_p is the p-Laplacian operator, λ is a positive parameter, $\alpha \in L^1(\Omega)$ is a non-zero potential, and $f : [0, +\infty[\to \mathbb{R}$ is a continuous function with $f(0) = 0$.

Problems of the type (P_λ) have been the object of intensive investigations in the recent years, see [1–8], and references therein. Many of the aforementioned contributions guarantee the existence of *at least two* non-trivial weak solutions of (P_λ) for $\lambda > 0$ large enough where the key geometric assumptions on the nonlinear term F, where $F : [0, +\infty[\to \mathbb{R}$ is the primitive of f, that is $F(s) = \int_0^s f(t)dt$ for every $s \geq 0$, can be summarized as

$$\begin{cases} \sup\limits_{[0,+\infty[} F > 0; \\ \limsup\limits_{s \to 0^+} \dfrac{F(s)}{s^p} \leq 0 \quad \text{and} \quad \limsup\limits_{s \to +\infty} \dfrac{F(s)}{s^p} \leq 0. \end{cases} \tag{1.1}$$

In order to obtain the aforementioned multiplicity results, various variational approaches are exploited; for instance, Morse theory [5,6], the mountain pass theorem and Ricceri-type three critical points results [1–4,7,9].

Notice that under (1.1) one can have even an exact multiplicity result for (P_λ). To see this, let $p = 2$, $n = 1$, $\Omega = I \subset \mathbb{R}$ be a large interval, $\alpha = 1$, and $f : [0, +\infty[\to \mathbb{R}$ defined by $f(s) = s(s - a)(1 - s)_+$ with $0 < a < 1/2$; here, $t_+ = \max(0, t)$. It is clear that F verifies (1.1). Moreover, via a bifurcation argument, Wei [10] proved that there exists $\lambda_0 > 0$ such that for

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all $0 < \lambda < \lambda_0$ problem (P_λ) has no positive solution, it has exactly one positive solution for $\lambda = \lambda_0$, and *exactly two* positive solutions for $\lambda > \lambda_0$; see also [11].

The main purpose of the present paper is to guarantee the existence of *at least three* non-zero, non-negative weak solutions for (P_λ) for certain values of $\lambda > 0$ when (1.1) holds. According to the above exact multiplicity result, our aim requires more specific assumptions both on f (or F) and α. In order to state our main result, we introduce the notation

$$k_\infty := \frac{n^{\frac{-1}{p}}}{\sqrt{\pi}}\left[\Gamma\left(1+\frac{n}{2}\right)\right]^{\frac{1}{n}}\left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} m(\Omega)^{\frac{1}{n}-\frac{1}{p}}, \tag{1.2}$$

where Γ denotes the Euler Gamma-function.

Our main result reads as follows:

Theorem 1.1. *Let $p > n$, $\alpha \in L^1(\Omega)$ be a non-negative, non-zero function with compact support K. Assume that*

(i) $S_F := \sup_{[0,+\infty[} F < +\infty$;
(ii) $\limsup_{s\to 0^+} \frac{F(s)}{s^p} \le 0$.

Moreover, there exists $c > 0$ such that

(iii) $F(c) = \max_{[0,k_\infty (pS_F \|\alpha\|_{L^1})^{\frac{1}{p}}]} F < S_F$;
(iv) $\frac{F(c)}{c^p} > \frac{m(\Omega\setminus K)}{p\,\mathrm{dist}(K,\partial\Omega)^p \|\alpha\|_{L^1}}$.

Then, the following statements hold:

(a) *For every $\lambda > 1$, problem (P_λ) has at least two non-zero, non-negative weak solutions.*
(b) *There exists $\hat{\lambda} > 1$ such that problem $(P_{\hat{\lambda}})$ has at least three non-zero, non-negative weak solutions.*

Before proving Theorem 1.1 some remarks are in order.

Remark 1.1. (a) Under the assumptions of Theorem 1.1, one can prove the existence of two non-zero weak solutions for (P_λ) for enough large values of $\lambda > 0$; the first one is the global minimum of the energy functional associated with (P_λ) with negative energy-level, while the second one is a mountain-pass type solution with positive energy-level. A much precise conclusion can be deduced as follows. Since (i),(ii) and (iv) imply (1.1), a suitable choice in [9] guarantees the existence of at least two non-zero weak solutions for (P_λ) for every $\lambda > \lambda_0$, where

$$\lambda_0 = \inf\left\{\frac{\int_\Omega |\nabla u|^p}{p\int_K \alpha(x)F(u(x))dx} : u \in W_0^{1,p}(\Omega),\ \int_K \alpha(x)F(u(x))dx > 0\right\}. \tag{1.3}$$

A simple estimate by means of a suitable truncation function and assumption (iv) show that

$$\lambda_0 < \frac{c^p m(\Omega\setminus K)}{pF(c)\mathrm{dist}(K,\partial\Omega)^p\|\alpha\|_{L^1}} < 1,$$

which concludes the proof of (a) in Theorem 1.1; for details see (3.6). Even more, under these assumptions, Ricceri's result (see [9]) provides a *stability* of problem (P_λ) with respect to any small nonlinear perturbation whenever $\lambda > \lambda_0$. However, for $\lambda > 0$ *small* enough, problem (P_λ) has usually only the trivial solution. Example 3.1 supports this fact as well.

(b) Assumption (iii) requires that the function F has a local maximum $c > 0$ on a quite large set whose size depends on the function F itself, namely, on the interval $I_F := [0, k_\infty(pS_F\|\alpha\|_{L^1})^{\frac{1}{p}}]$. Note that a simple estimate together with hypothesis (iv) shows that c belongs to the interval I_F. In view of the above discussion, the technical assumption (iii) is behind on the existence of a third non-zero weak solution for (P_λ).

Remark 1.2. Note that in Theorem 1.1 we are able to prove the existence of a single value of $\hat{\lambda} > 1$ such that problem $(P_{\hat{\lambda}})$ has at least three non-zero, non-negative weak solutions. A challenging problem is to know if this phenomenon is stable/unstable with respect to the parameter λ; namely, to confirm/infirm the existence of certain functions f satisfying all the assumptions of Theorem 1.1 such that problem (P_λ) has exactly two non-zero weak solutions for $\lambda \in]1, +\infty[\setminus\{\hat{\lambda}\}$ and at least three solutions for $\lambda = \hat{\lambda}$.

Remark 1.3. Taking into account the special character of the function α (i.e., α has a compact support K in Ω), we could expect to construct in a trivial way some weak solutions for (P_λ) via p-harmonic functions. The reason is the following; for simplicity, let us consider the case when $\Omega = B(0,R)$ and $K = \overline{B}(0,r)$ for some $0 < r < R$. Due to (iii), the nonlinearity f attains the zero value at least in two points (c being one of them since it is a local maximum for F). Let us denote such an element by $c > 0$. A simple calculation shows that the function $\tilde{u}_c \in W_0^{1,p}(B(0,R))$ defined by

$$\tilde{u}_c(x) = \begin{cases} c & \text{if } x \in K = \overline{B}(0,r), \\ c\dfrac{|x|^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}}{r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}} & \text{if } x \in B(0,R)\setminus K, \end{cases}$$