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Large Eddy Simulation for turbulent flows with critical regularization

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ABSTRACT

In this paper, we establish the existence of a unique "regular" weak solution to the Large Eddy Simulation (LES) models of turbulence with critical regularization. We first consider the critical LES for the Navier–Stokes equations and we show that its solution converges to a solution of the Navier–Stokes equations as the averaging radii converge to zero. Then we extend the study to the critical LES for magnetohydrodynamic equations.

1. Introduction

Let us consider the Navier–Stokes equations in a three dimensional torus $\mathbb{T}_3$,

$$\mathrm{div}\,\boldsymbol{v} = 0, \tag{1.1}$$

$$\boldsymbol{v}_{,t} + \mathrm{div}\,(\boldsymbol{v}\otimes\boldsymbol{v}) - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f}, \tag{1.2}$$

subject to $\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x})$. Here, $\boldsymbol{v}$ is the fluid velocity field, p is the pressure, $\boldsymbol{f}$ is the external body force, ν stands for the viscosity.

Eqs. (1.1) and (1.2) are known to be the idealized physical model to compute Newtonian fluid flows. They are also known to be unstable in numerical simulations when the Reynolds number is high, thus when the flow is turbulent. Therefore, numerical turbulent models are needed for real simulations of turbulent flows. In many practical applications, knowing the mean characteristics of the flow by averaging techniques is sufficient. However, averaging the nonlinear term in NSE leads to the well-known closure problem. To be more precise, if $\overline{\boldsymbol{v}}$ denotes the filtered/averaged velocity field then the Reynolds averaged Navier-Stokes (RANS) equations

$$\overline{\boldsymbol{v}}_{,t} + \mathrm{div}\,(\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}) - \nu\Delta\overline{\boldsymbol{v}} + \nabla\overline{p} + \mathrm{div}\,\mathcal{R}(\boldsymbol{v},\boldsymbol{v}) = \overline{\boldsymbol{f}}, \tag{1.3}$$

where $\mathcal{R}(\boldsymbol{v},\boldsymbol{v}) = \overline{\boldsymbol{v}\otimes\boldsymbol{v}} - \overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}$, the Reynolds stress tensor, is not closed because we cannot write it in terms of $\overline{\boldsymbol{v}}$ alone. The main essence of turbulence modeling is to derive simplified, reliable and computationally realizable closure models. In [1,2] Layton and Lewandowski suggested an approximation of the Reynolds stress tensor given by

$$\mathcal{R}(\boldsymbol{v},\boldsymbol{v}) = \overline{\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}} - \overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}. \tag{1.4}$$

This is an equivalent form to the approximation

$$\mathrm{div}\,(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}) \approx \mathrm{div}\,(\overline{\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}}). \tag{1.5}$$

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Hence Layton and Lewandowski studied the following Large Scale Model considered as a Large Eddy Simulation (LES) model:

$$\operatorname{div} \boldsymbol{w} = 0, \tag{1.6}$$

$$\boldsymbol{w}_{,t} + \operatorname{div}\,(\overline{\boldsymbol{w} \otimes \boldsymbol{w}}) - \nu \Delta \boldsymbol{w} + \nabla q = \overline{\boldsymbol{f}}, \tag{1.7}$$

considered in $(0, T) \times \mathbb{T}_3$ and subject to $\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{w}_0(\boldsymbol{x}) = \overline{\boldsymbol{v}_0}$ and periodic boundary conditions with mean value equal to zero. Where they denoted $(\boldsymbol{w}, q)$ as the approximation of $(\overline{\boldsymbol{v}}, \overline{p})$.

The averaging operator chosen in (1.7) is a differential filter, [3–5,1,6,7], that commutes with differentiation under periodic boundary conditions and is defined as follows. Let $\alpha > 0$, given a periodic function $\varphi \in L^2(\mathbb{T}_3)$, define its average $\overline{\varphi}$ to be the unique solution of

$$-\alpha^2 \Delta \overline{\varphi} + \overline{\varphi} = \varphi, \tag{1.8}$$

The main goal in using such a model is to filter eddies of scale less than the numerical grid size α in numerical simulations. The Laplacian in the above expression has a smoothing effect and this allow us to prove existence and uniqueness of the solution. In some cases, the use of the smoothing effect of the Laplacian can be unnecessary. However, we may use other filters, as the top-hat filter [8,9] which are not smoothing, or as differential filter with fractional order Laplace operator [10–12]. Moreover, Layton and Neda [13] observed by using the classical dimensional analysis arguments of Kolmogorov coupled with precise mathematical knowledge of the model's kinetic energy balance that the energy spectra of the LES model (1.6) and (1.7) should scale as

$$E(\boldsymbol{k}) \cong \epsilon_\alpha^{\frac{2}{3}} |\boldsymbol{k}|^{-\frac{5}{3}}, \quad \text{for } \alpha \le \frac{1}{|\boldsymbol{k}|}, \tag{1.9}$$

$$E(\boldsymbol{k}) \cong \epsilon_\alpha^{\frac{2}{3}} \alpha^{-2} |\boldsymbol{k}|^{-\frac{11}{3}}, \quad \text{for } \alpha \ge \frac{1}{|\boldsymbol{k}|}, \tag{1.10}$$

where ϵ_α is the time averaged energy dissipation rate of the model's solution given by

$$\epsilon_\alpha = \left\langle \frac{\nu}{L^3} \left(\|\boldsymbol{w}\|_2^2 + \alpha^2 \|\boldsymbol{w}\|_{1,2}^2 \right) \right\rangle. \tag{1.11}$$

Here, $\|\boldsymbol{w}\|_{1,2}^2$ represents the $W^{1,2}(\mathbb{T}_3)^3$ semi-norm which is given by

$$\|\boldsymbol{w}\|_{1,2}^2 = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} |\boldsymbol{k}|^2 \, |\widehat{\boldsymbol{w}}(\boldsymbol{k})|^2.$$

Thus the application of a smooth filter strongly affects the shape of the energy spectra. In particular, we observe that there are two different power laws for the energy cascade. For wave numbers $\boldsymbol{k}$ such that $|\boldsymbol{k}| \le \frac{1}{\alpha}$ we obtain the usual $|\boldsymbol{k}|^{\frac{-5}{3}}$ Kolmogorov power law. This implies that the large scale statistics of the flow of size greater than the length scale α are consistent with the Kolmogorov theory for 3D turbulent flows. On the other hand, for $|\boldsymbol{k}| \ge \frac{1}{\alpha}$ we obtain a steeper power law. This implies a faster decay of energy in comparison to direct numerical simulation (DNS), which suggests, in terms of numerical simulation, a smaller resolution requirement in computing turbulent flows.

For a general overview of LES models, the readers are referred to Berselli et al. [8] and references cited therein. Notice that the Layton–Lewandowski model (1.6) and (1.7) differs from the one introduced by Bardina et al. [14] where the following approximation of the Reynolds stress tensor is used:

$$\mathcal{R}(\boldsymbol{v}, \boldsymbol{v}) = \overline{\overline{\boldsymbol{v}} \otimes \overline{\boldsymbol{v}}} - \overline{\overline{\boldsymbol{v}}} \otimes \overline{\overline{\boldsymbol{v}}}. \tag{1.12}$$

In [1,2] Layton and Lewandowski have proved that (1.6) and (1.7) have a unique regular solution. They have also shown that there exists at least a sequence α_j which converges to zero and such that the sequence $(\boldsymbol{w}_{\alpha_j}, q_{\alpha_j})$ converges to a distributional solution $(\boldsymbol{v}, p)$ of the Navier–Stokes equations.

We remark that many of these results established in the above cited papers have been extended to the following three dimensional magnetohydrodynamic equations (MHD):

$$\operatorname{div} \mathcal{B} = \operatorname{div} \boldsymbol{v} = 0 \tag{1.13}$$

$$\partial_t \boldsymbol{v} - \nu_1 \Delta \boldsymbol{v} + \operatorname{div}\,(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}\,(\mathcal{B} \otimes \mathcal{B}) + \nabla p = 0, \tag{1.14}$$

$$\partial_t \mathcal{B} - \nu_2 \Delta \mathcal{B} + \operatorname{div}\,(\boldsymbol{v} \otimes \mathcal{B}) - \operatorname{div}\,(\mathcal{B} \otimes \boldsymbol{v}) = 0, \tag{1.15}$$

$$\int_{\mathbb{T}_3} \mathcal{B}\, d\boldsymbol{x} = \int_{\mathbb{T}_3} \boldsymbol{v}\, d\boldsymbol{x} = 0 \tag{1.16}$$

$$\mathcal{B}(0) = \mathcal{B}_0, \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{1.17}$$

Here $\boldsymbol{v}$ is the fluid velocity field, p is the fluid pressure, $\mathcal{B}$ is the magnetic field, and $\boldsymbol{v}_0$ and $\mathcal{B}_0$ are the corresponding initial data. The interested readers are referred to [15,16] and references cited therein.