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Moments of discrete measures with dense jumps induced by β-expansions

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ABSTRACT

Let $\beta > 1$. Through an appeal to β-expansions we define a strictly increasing and left-continuous function μ_β on $[0, 1]$. Then μ_β turns out to be a pure jump distribution. In other words, its associated Lebesgue–Stieltjes measure is discrete, i.e., a summation of point masses. The present note studies the moment of this discrete measure and its asymptotics.

1. Introduction

A numeration is a systematic means of representing numbers by finite or infinite words. Considering two different numerations at once, we have so far obtained some important (counter-)examples for analysis. Just as the ternary expansions together with the binary expansions produce the Cantor function, Minkowski's $?(x)$ function is obtained via the (regular) continued fraction and the alternated binary expansions [1]. Both functions are quite exceptional because they are *singular functions*, i.e., their derivatives vanish almost everywhere in the Lebesgue measure sense. They are continuous monotone functions. But Minkowski's $?(x)$ function is, unlike the Cantor function, strictly increasing. So the singularity of Minkowski's function was even more curious when Denjoy [2] and later Salem [3] proved it. Another strictly increasing singular function was also realized by Riesz and Sz.-Nágy [4]. Recently the Riesz–Nágy function has been generalized in [5], where it was shown that its differentiability at x rests upon the normality of x to base 2. On the other hand, the differentiability of $?(x)$ is tightly related to the continued fraction of x [6,7]. A more bald connection between differentiability and Diophantine property was demonstrated by a singular function $\Delta(x)$ in [8]. Although $\Delta(x)$ is discontinuous at every rational $x > 0$, it was proved that $\Delta(x)$ is differentiable unless x is extremely well approximable by rationals. In particular, at any non-Liouville irrational $x > 0$, the function Δ is differentiable and $\Delta'(x) = 0$, whereas, at x whose continued fraction is such as $x = [0; 1, 2^2, 3^{3^3}, 4^{4^{4^4}}, \ldots]$, Δ is not differentiable. The function Δ has many properties in common with *saltus functions* considered in [9]. Very recently, Δ has been generalized to a two-variable function $\Xi : [0, \infty) \times [1, \infty) \to \mathbb{R}$ in [10], which is discontinuous on a dense set but total differentiable almost everywhere. Via Ξ, we constructed a two-variable singular function, that is, a non-constant continuous function that is locally constant on a set of full measure. The function Ξ is our main topic, and hence will be explicitly defined below.

The moments of the above mentioned distributions have been well studied in diverse contexts. For the moments of the Cantor function, see a survey [11] and the bibliography therein. And Alkauskas has been studying the moments of Minkowski's $?(x)$ function in a series of papers [12–14]. The moments of the generalized Riesz–Nágy distribution were computed by Baek [15] in terms of a recurrence relation. For a fixed $\beta > 1$, the present paper considers a distribution

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$\mu_\beta(x) := \Xi(x, \beta)$ on the interval [0, 1], and analyzes its moments

$$M_m := \int_0^1 x^m d\mu_\beta, \quad m = 0, 1, 2, \ldots.$$

The asymptotics of M_m are also explored as m tends to infinity.

We will note below that, with being discontinuous at every rational, the function $\mu_\beta(x)$ is singular. And we will prove that it is a pure jump distribution. In other words, the Lebesgue–Stieltjes measure induced by this distribution is discrete, and point masses are distributed over the whole rationals in the interval [0, 1]. The moments of discrete measures with dense jumps were also considered in [16,17] to investigate the asymptotic behavior of the coefficients of orthogonal polynomials.

In Section 2, we define the distribution μ_β with some preliminaries. Section 3 derives a closed-form formula for the moments of μ_β, and with this formula in hand, their asymptotics are investigated in Section 4. Some concrete examples are also presented in the last section.

2. Definition and property of μ_β

Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ be the floor and ceiling functions respectively. We denote by $\mathbb{N}$ the set of nonnegative integers. For real $\alpha, \rho \in [0, 1]$, two functions $s_{\alpha,\rho}, s'_{\alpha,\rho} : \mathbb{N} \to \mathbb{N}$ are defined by

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$
$$s'_{\alpha,\rho}(n) := \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil,$$

which yield infinite words $s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots$ and $s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots$. Note that $s_{\alpha,\rho}(n) = s'_{\alpha,\rho}(n)$ provided that neither $\alpha n + \rho$ nor $\alpha(n+1) + \rho$ is equal to an integer. Actually, if $\alpha n + \rho$ is an integer for $\alpha \neq 0, 1$, then

$$s_{\alpha,\rho}(n) = 0, \qquad s'_{\alpha,\rho}(n) = 1 \quad \text{and} \quad s_{\alpha,\rho}(n-1) = 1, \qquad s'_{\alpha,\rho}(n-1) = 0.$$

The word $s_{\alpha,\rho}$ (resp. $s'_{\alpha,\rho}$) is termed a *lower* (resp. *upper*) *mechanical word* with *slope* α and *intercept* ρ. It readily follows that $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are binary or unary words over the alphabet $A := \{0, 1\}$. For $\alpha = 0$ or 1, $s_{\alpha,\rho} = s'_{\alpha,\rho} = \alpha^\omega := \alpha\alpha\cdots$. If α is neither 0 nor 1, then both 0 and 1 appear in $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$. This type of words has been a flourishing topic in combinatorics on words. The interested readers are referred to [18].

For $\beta > 1$, a function $(\cdot)_\beta$ is defined to send each infinite word $a_0a_1\cdots \in A^{\mathbb{N}}$ to a real number $\sum_{i=0}^\infty a_i/\beta^{i+1}$. If the word $a_0a_1\cdots$ satisfies some lexicographical condition, then we say that $a_0a_1\cdots$ is the β-expansion of $\sum_{i=0}^\infty a_i/\beta^{i+1}$ [19]. Since Rényi [20] and Parry [19] did it, numerous mathematicians studied β-expansions from an ergodic point of view. Now we define the function $\Xi : [0, 1] \times (1, \infty) \to \mathbb{R}$ by

$$\Xi(\alpha, \beta) := (s'_{\alpha,0})_\beta.$$

Though α is restricted here to the unit interval [0, 1], it can be extended to $[0, \infty)$ as in [10]. Note that $s'_{\alpha,0}$ fulfills Parry's lexicographical condition so that $s'_{\alpha,0}$ is a β-expansion of 1 for some $\beta > 1$ (cf. [21, Proposition 3.2]). It is also worthwhile mentioning that the map Ξ was alternatively embodied, in [21], by β-transformations. On the other hand, via the level curve $\Xi(\alpha, \beta) = 1$ in [22], the author completely characterized generalized baker's transformations in a natural sense.

Instead of $(s'_{\alpha,0})_\beta$ in the definition of Ξ, we can also study the case of $(s_{\alpha,0})_\beta$ with little modification. But, for a fixed $\rho \neq 0$, it is difficult to analyze $(s_{\alpha,\rho})_\beta$ and $(s'_{\alpha,\rho})_\beta$ as functions in α and β. Their differentiabilities are necessarily accompanied by inhomogeneous Diophantine approximations if $\rho \notin \alpha\mathbb{Z} + \mathbb{Z}$, which are much more involved than homogeneous ones.

Let us set a family of functions $\mu_\beta : [0, 1] \to \mathbb{R}$ with a parameter $\beta > 1$ by

$$\mu_\beta(x) := \Xi(x, \beta).$$

From now on, we assume that $\beta > 1$ is fixed unless stated explicitly. The next lemma is immediate to observe.

Lemma 2.1. *Let $\alpha_1, \alpha_2 \in [0, 1]$. The following are equivalent.*

(a) $\alpha_1 < \alpha_2$.
(b) $s'_{\alpha_1,0} < s'_{\alpha_2,0}$ *lexicographically.*
(c) $\mu_\beta(\alpha_1) < \mu_\beta(\alpha_2)$.

Proof. The equivalence between (a) and (b) is well-known. See, e.g., [18]. If $\alpha_1 < \alpha_2$, then

$$\begin{aligned} 0 \le \sum_{n=0}^\infty \frac{\lceil \alpha_2 n \rceil - \lceil \alpha_1 n \rceil}{\beta^{n+1}} &= \frac{1}{\beta} \sum_{n=1}^\infty \frac{\lceil \alpha_2 n \rceil - \lceil \alpha_1 n \rceil}{\beta^n} \\ &= \frac{1}{\beta} \sum_{n=0}^\infty \frac{\lceil \alpha_2 (n+1) \rceil - \lceil \alpha_1 (n+1) \rceil}{\beta^{n+1}} \\ &\le \sum_{n=0}^\infty \frac{\lceil \alpha_2 (n+1) \rceil - \lceil \alpha_1 (n+1) \rceil}{\beta^{n+1}}, \end{aligned}$$