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Normal forms of necessary conditions for dynamic optimization problems with pathwise inequality constraints

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ABSTRACT

There has been a longstanding interest in deriving conditions under which dynamic optimization problems are normal, that is, the necessary conditions of optimality (NCO) can be written with a nonzero multiplier associated with the objective function. This paper builds upon previous results on nondegenerate NCO for trajectory constrained optimal control problems to provide even stronger, normal forms of the conditions. The NCO developed may address problems with nonsmooth, less regular data. The particular case of calculus of variations problems is here explored to show a favorable comparison with existent results.

1. Introduction

In this paper, we study Necessary Conditions of Optimality (NCO) for Dynamic Optimization Problems with pathwise inequality constraints. In particular, we are interested in normal forms of the NCO, i.e., forms in which the scalar multiplier associated with the objective function – here called λ – is nonzero. The normal forms of the NCO are guaranteed to supply non-trivial information, in the sense that they guarantee that the objective function is taken into account when selecting candidates to optimal processes.

Many important applications of NCO would benefit or even require normal forms. In engineering applications or in decision making contexts, the NCO are used to select a candidate (or a small number of candidates) for optimal solution. If we do not guarantee normality and allow $\lambda = 0$, then the NCO identify a set of candidates in which the objective function is not used in the selection, and such an identified set is typically too large. This is even more critical in applications where the NCO are used to find a solution without human intervention (e.g. synthesis of controls for autonomous vehicles), and thus we have to guarantee that the NCO remain informative.

Normal forms of NCO are also important in establishing results on the regularity properties of optimal solutions and to establish second-order conditions. In most results of such nature, the possibility of selecting $\lambda \neq 0$ has to be assumed (e.g. [1–5]) or conditions are imposed so as to guarantee that the system of first-order conditions is normal (e.g. [6,7]).

The importance of studying normal forms of NCO is well illustrated in the history of Mathematical Programming [8,9]. The Kuhn–Tucker conditions [10], one of the most cited results in optimization, are a strengthened, nondegenerate version of some earlier conditions, now less known, of Fritz John [11].

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There has been a growing interest and literature on strengthened forms of NCO for Optimal Control Problems (OCP), reporting both nondegenerate and normal forms of the maximum principle (MP). (See e.g. [12] for what appears to be the first result on the subject, the recent works [13,14] and references therein, as well as [15] which provides references to an extensive Russian literature on the subject.) The normality results reported in literature require different degrees of regularity on the problem data [16–21]. Requiring very little regularity on the data, we can find strengthened NCO in [22] which, although not ensuring normality, are able to avoid certain sets of degenerate multipliers. Building upon the nondegeneracy results in [22], we develop here an even stronger form of NCO: a normal form. An advantage of our result comparing with similar results in literature is the fact that it addresses problems with less regular, nonsmooth data. However, the additional hypotheses under which our result is valid, known as constraint qualification (CQ), involve the optimal control which we do not know in advance, and consequently, in general, it is not so easy to verify whether the CQ is satisfied for the problem we have in hands. Nevertheless, in some cases, the conditions we propose compare favorably with existing results. One such case is the application of our result to calculus of variations problems (CVP). We study normality of NCO for CVP as a consequence of the results on normality of NCO for OCP here developed. The special structure of CVP permits the derivation of CQ that are much easier to verify than in the optimal control case. The conditions thereby obtained generalize a result in [16] to the nonsmooth case.

This paper is organized as follows. In a brief Preliminaries section, we provide some of the concepts and notation that are used throughout the paper. Section 3 describes the context of our results: optimal control problems with state constraints and the nonsmooth maximum principle that is to be strengthened in later sections. We also describe the case of CVP with inequality constraints and its necessary conditions of optimality. Section 4 provides a main result of this paper: a normal form of NCO valid under a suitable constraint qualification. In Section 5, we apply the previous result to a CVP and deduce CQs which are specific for this problem and have the advantage that they are easy to verify. In Section 6 we compare the results obtained in the previous section with other results when applied to CVPs. Finally, in Sections 7 and 8, we prove the main results and lemmas of this paper.

2. Preliminaries

Throughout, $\mathbb{B}$ denotes the *closed unit ball*, co S denotes the *convex hull* of a set S, supp$\{\mu\}$ denotes the support of measure μ, and $\delta_{\{0\}}$ denotes the Dirac unit measure concentrated at 0. We also make reference to the space $W^{1,1}$ of absolutely continuous functions, C^* the dual space of continuous functions, and $C^{1,1}$ the space of functions which are continuously differentiable with locally Lipschitz continuous derivatives.

The *limiting normal cone* of a closed set $C \subset \mathbb{R}^n$ at $\bar{x} \in C$ is defined to be

$$N_C^L(\bar{x}) := \{\eta \in \mathbb{R}^n : \exists \text{ sequences } \{M_i\} \in \mathbb{R}^+, x_i \to \bar{x}, \eta_i \to \eta \text{ such that}$$
$$x_i \in C \text{ and } \eta_i \cdot (y - x_i) \le M_i \|y - x_i\|^2 \text{ for all } y \in \mathbb{R}^n, i = 1, 2, \ldots\}.$$

Given a lower semicontinuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ the *limiting subdifferential* of f at a point $\bar{x} \in \mathbb{R}^n$ such that $f(x) < +\infty$ is the set

$$\partial^L f(\bar{x}) = \{\eta \in \mathbb{R}^n : (\eta, -1) \in N^L_{epif}(\bar{x}, f(\bar{x}))\};$$

where $epi f := \{(x, \alpha) : \alpha \ge f(x)\}$. We also make use of the *hybrid partial subdifferential* of h in the x-variable defined as

$$\partial_x^> h(t, x) := \text{co}\{\xi : \text{there exist } (t_i, x_i) \to (t, x) \text{ s.t.}$$
$$h(t_i, x_i) > 0, h(t_i, x_i) \to h(t, x), \text{ and } h_x(t_i, x_i) \to \xi\}.$$

We refer to [23–25] for further concepts of nonsmooth analysis and optimal control. See also [26] for a review using a notation similar to the one used here.

3. Context

Consider the fixed left-endpoint Optimal Control Problem (OCP) with inequality state constraints:

$$(\text{OCP}_1) \begin{cases} \text{Minimize} & g(x(1)) \\ \text{subject to} & \\ & \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0, 1] \\ & x(0) = x_0 \\ & u(t) \in \Omega(t) \quad \text{a.e. } t \in [0, 1] \\ & h(t, x(t)) \le 0 \quad \text{for all } t \in [0, 1]. \end{cases}$$

The data for this problem comprise functions $g : \mathbb{R}^n \mapsto \mathbb{R}$, $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$, $h : [0, 1] \times \mathbb{R}^n \mapsto \mathbb{R}$, an initial state $x_0 \in \mathbb{R}^n$, and a multifunction $\Omega : [0, 1] \rightrightarrows \mathbb{R}^m$.

The set of *control functions* for (OCP_1), denoted $\mathcal{U}$, is the set of measurable functions $u : [0, 1] \to \mathbb{R}^m$ such that $u(t) \in \Omega(t)$ a.e. $t \in [0, 1]$. A *state trajectory* is an absolutely continuous function which satisfies the differential equation in the constraints for some control function u. The domain of the above optimization problem is the set of *admissible processes*,