J. Math. Anal. Appl. 399 (2013) 38–49

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Remarks on the multiplier operators associated with a cylindrical distance function

Sunghun Choi [a], Sunggeum Hong [b,*], Sanghyuk Lee [c]

[a] Department of Mathematics, Postech, Pohang 790-784, Republic of Korea
[b] Department of Mathematics, Chosun University, Gwangju 501-759, Republic of Korea
[c] School of Mathematical Sciences, Seoul National University, Seoul 151-742, Republic of Korea

ARTICLE INFO

Article history:
Received 19 March 2012
Available online 29 September 2012
Submitted by Eero Saksman

Keywords:
Cylinder multiplier
Maximal estimates

ABSTRACT

In this note, we consider L^p and maximal L^p estimates for the generalized Riesz means which are associated with the cylindrical distance function $\rho(\xi) = \max\{|\xi'|, |\xi_{d+1}|\}$, $(\xi', \xi_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$. We prove these estimates up to the currently known range of the spherical Bochner–Riesz and its maximal operators. This is done by establishing implications between the corresponding estimates for the spherical Bochner–Riesz and the cylindrical multiplier operators.

1. Introduction and statement of results

In this paper we consider multiplier operators associated with a rough distance function, which are known as the cylinder multiplier operators. More precisely, we define a distance function ρ by

$$\rho(\xi) = \max\{|\xi'|, |\xi_{d+1}|\}, \quad \xi = (\xi', \xi_{d+1}) \in \mathbb{R}^d \times \mathbb{R}.$$

The generalized Riesz means of order $\alpha \geq 0$ which are associated with ρ is defined by

$$\widehat{S_t^\alpha f}(\xi) = \left(1 - \frac{\rho(\xi)}{t}\right)_+^\alpha \widehat{f}(\xi).$$

Here $a_+^\alpha = a^\alpha$ for $a > 0$ and $a_+^\alpha = 0$ otherwise. In connection with the convergence of $S_t f \to f$ in L^p as $t \to \infty$, the inequality

$$\|S_1^\alpha f\|_{L^p(\mathbb{R}^{d+1})} \leq C\|f\|_{L^p(\mathbb{R}^{d+1})} \tag{1.1}$$

has been studied by some authors [1,2].

As it was shown in [2], L^p boundedness of S_1^α is closely related to those of the spherical Bochner–Riesz and the cone multiplier operators. For $1 \leq p \leq \infty$, let

$$\alpha(p) = \max\left\{d\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right\}$$

* Corresponding author.
E-mail addresses: zalhagge@postech.ac.kr (S. Choi), skhong@chosun.ac.kr (S. Hong), shklee@snu.ac.kr (S. Lee).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.09.048

be the critical exponent for L^p boundedness of Bochner–Riesz operators in $\mathbb{R}^d$ and the cone multiplier operator in $\mathbb{R}^{d+1}$. We now set

$$\widehat{T_t^\alpha g}(\xi') = \left(1 - \frac{|\xi'|^2}{t^2}\right)_+^\alpha \widehat{g}(\xi'), \quad \xi' \in \mathbb{R}^d.$$

The conjecture which is known as Bochner–Riesz conjecture is that for $1 \le p \le \infty$

$$\|T_1^\alpha g\|_{L^p(\mathbb{R}^d)} \le C\|g\|_{L^p(\mathbb{R}^d)} \tag{1.2}$$

if and only if $\alpha > \alpha(p)$. When $d = 2$, it was verified by Carleson and Sjölin [3]. In higher dimensions it is still open and some partial results are known. Indeed, L^p boundedness on the range $(2d+2)/(d-1) \le p \le \infty$ and $1 \le p \le (2d+2)/(d+1)$ is due to the sharp L^2 restriction estimate [4] and the argument of Stein (for example, see, [5, pp. 422–423]). Beyond these results, progresses have been made (see [6–9] and references therein). Most recent results are due to the third author [8] (also see [10]) when $d = 3, 4$, and Bourgain and Guth [11] when $d \ge 5$.

By de Leeuw's restriction theorem the boundedness of S_1^α on $L^p(\mathbb{R}^{d+1})$ implies that of the Bochner–Riesz operator of the same order on $L^p(\mathbb{R}^d)$. From the known necessary condition for (1.2), it follows that S_1^α is bounded on L^p only if $\alpha > \alpha(p)$. When $d \ge 3$ there is an additional necessary condition that $\frac{2d}{d+3} < p < \frac{2d}{d-3}$. It is due to the fact that near the surface $|\xi'| = \xi_{d+1}$, the multiplier $\left(1 - \frac{\rho(\xi)}{t}\right)_+^\alpha$ behaves similarly as the cone multiplier of order 1. So it was conjectured [1,2] that (1.1) holds if and only if $\alpha > \alpha(p)$ and $\frac{2d}{d+3} < p < \frac{2d}{d-3}$ when $d \ge 3$ and $1 < p < \infty$ when $d = 2$. In [2] the problem was settled when $d = 2$, and some partial results were obtained when $d \ge 3$. For further progress in higher dimensions, one should improve boundedness of Bochner–Riesz operators. However, thanks to recent progress on the problem of the cone multiplier [12] (also see [13,14]), it is possible to show that L^p boundedness of S_1^α is equivalent to that of T_1^α.

Theorem 1.1. *Let $1 < p < \infty$ when $d = 2$, and let $\frac{2d}{d+3} < p < \frac{2d}{d-3}$ when $d \ge 3$. (1.2) holds for $\alpha > \alpha(p)$ if and only if (1.1) holds for $\alpha > \alpha(p)$.*

So this establishes L^p boundedness of the cylinder multiplier operators up to the currently known range of Bochner–Riesz operators. That is to say, (1.1) holds if $p_\circ \le p \le \infty$, $1 \le p \le p_\circ'$, and $\alpha > \alpha(p)$ where $p_\circ$ is given by

$$p_\circ = 2 + \frac{12}{4d-3-k} \quad \text{if } d \equiv k \ (\mathrm{mod}\ 3),\ k = -1, 0, 1.$$

Next we consider the maximal operator

$$S_*^\alpha f(x) = \sup_{t>0} |S_t^\alpha f(x)|.$$

In general, L^p estimate for $S_*^\alpha f$ has been of interest in connection with almost everywhere convergence of $S_t^\alpha f$ as $t \to \infty$ and it is also an obvious extension of (1.1). The same problems for the maximal Bochner–Riesz operator $T_*^\alpha g(x') = \sup_{t>0} |T_t^\alpha g(x')|$ have been studied in [15,16,8] and it is conjectured that for $2 \le p \le \infty$

$$\|T_*^\alpha g\|_{L^p(\mathbb{R}^d)} \le C\|g\|_{L^p(\mathbb{R}^d)} \tag{1.3}$$

holds if and only if $\alpha > \alpha(p)$. This was settled by Carbery [15] when $d = 2$. Partial results are known [16,8] when $d \ge 3$ so that the conjecture is verified for $p \ge 2 + \frac{4}{d}$. It seems possible that recent progress [11] leads to further improvement. On the contrary, for $p < 2$ Tao [17] showed that the L^p boundedness of T_*^α is different from that of T_1^α, and when $d = 2$ he [18] also obtained some improvement upon the classical result [19].

It is natural to expect that the maximal estimate

$$\|S_*^\alpha f\|_{L^p(\mathbb{R}^{d+1})} \le C\|f\|_{L^p(\mathbb{R}^{d+1})} \tag{1.4}$$

holds provided that $\alpha > \alpha(p)$, and $2 \le p \le \infty$ when $d = 2$ and $2 \le p < \frac{2d}{d-3}$ when $d \ge 3$. For $0 < p < 1$ the boundedness of S_*^α from H^p to $L^{p,\infty}$ was shown in [20]. But, as far as the authors know, nothing is known about (1.4) for $p \ge 1$. In what follows we shall show that the similar implication also holds for the maximal estimates.

Theorem 1.2. *Let $2 \le p \le \infty$ when $d = 2$ and let $2 \le p < \frac{2d}{d-3}$ when $d \ge 3$. If (1.3) holds for $\alpha > \alpha(p)$, then (1.4) holds for $\alpha > \alpha(p)$.*

Hence, this establishes the boundedness of S_*^α up to that of currently known range of maximal Bochner–Riesz operators. So, (1.4) holds for $p > 2 + \frac{4}{d}$ (see [8]).