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On a Neumann problem with critical exponents and Hardy potentials

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ABSTRACT

In this paper, we consider the Neumann problem of the potential elliptic system with Hardy terms and critical exponents on a bounded domain. Using the critical point theory, we first find infinitely many solutions for the corresponding perturbed subcritical problem. Analyzing concentration points in the bubbles carefully, which appear in the approximate solutions, we establish the concentration estimates, and then obtain infinitely many solutions with positive energy by passing to a limit.

1. Introduction and main results

Let Ω be a smooth open bounded domain in R^N with $N \geq 3$, $0 \in \Omega$. We consider

$$\begin{cases} -\Delta u - t\dfrac{u}{|x|^2} + \lambda u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \dfrac{2p}{p+q}|u|^{p-2}u|v|^{q} & \text{in } \Omega, \\ -\Delta v - t\dfrac{v}{|x|^2} + \mu v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \dfrac{2q}{p+q}|u|^{p}|v|^{q-2}v & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda,\ \mu > 0$, $p + q < 2^*$ with $p, q > 1$, $\alpha,\ \beta > 1$ satisfy $\alpha + \beta = 2^*$, $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent, ν is the unit outward normal at the boundary $\partial\Omega$.

A pair of functions $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ is said to be a weak solution of problem (1.1) if it holds for any $\varphi = (\varphi_1, \varphi_2) \in H^1(\Omega) \times H^1(\Omega)$

$$\int_\Omega \left(\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 - t\frac{u\varphi_1}{|x|^2} - t\frac{v\varphi_2}{|x|^2} + \lambda u\varphi_1 + \mu v \varphi_2\right) dx - \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^{\beta}\varphi_1 dx$$
$$- \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^{\beta-2}v\varphi_2 dx - \frac{2p}{p+q}\int_\Omega |u|^{p-2}u|v|^{q} dx - \frac{2q}{p+q}\int_\Omega |u|^{p}|v|^{q-2}v dx = 0.$$

The corresponding energy functional of problem (1.1) is defined on $H^1(\Omega) \times H^1(\Omega)$ by

$$\begin{aligned} I_{\lambda,\mu}(u, v) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 - t\frac{|u|^2 + |v|^2}{|x|^2} + \lambda|u|^2 + \mu|v|^2\right) dx \\ - \frac{2}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^{\beta} dx - \frac{2}{p+q}\int_\Omega |u|^{p}|v|^{q} dx. \end{aligned} \tag{1.2}$$

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It is well known that the nontrivial solutions of problem (1.1) are equivalent to the nonzero critical points of $I_{\lambda,\mu}$ in $H^1(\Omega) \times H^1(\Omega)$. Moreover, every weak solution (u, v) of problem (1.1) satisfies that $u, v \in C^2(\Omega) \bigcap C^1(\overline{\Omega} \setminus \{0\})$ (see Remark 4 in [1]).

Set $D^{1,2}(R^N) = \{u \in L^{2^*}(R^N) \mid |\nabla u| \in L^2(R^N)\}$. For all $t \in [0, \bar{t}), \bar{t} = \left(\frac{N-2}{2}\right)^2$, define the constant $S_t := \inf\left\{\int_{R^N}\left(|\nabla u|^2 - t\frac{|u|^2}{|x|^2}\right)dx \Big| \int_{R^N}|u|^{2^*}dx = 1, \ \forall u \in D^{1,2}(R^N)\right\}$. From [2], S_t is independent of any $\Omega \subset R^N$ in the sense that if $S_t := \inf\left\{\int_\Omega\left(|\nabla u|^2 - t\frac{|u|^2}{|x|^2}\right)dx \Big| \int_\Omega |u|^{2^*}dx = 1, \ \forall u \in H_0^1(\Omega)\right\}$, then $S_t(\Omega) = S_t(R^N) = S_t$. Especially, S_0 is the best Sobolev constant defined by $S_0 := \inf\left\{\int_\Omega |\nabla u|^2 dx \Big| \int_\Omega |u|^{2^*}dx = 1, \ \forall u \in H_0^1(\Omega)\right\}$, which is achieved if and only if $\Omega = R^N$ by $U(x) = (N(N-2))^{\frac{N-2}{4}}(1+|x|^2)^{-\frac{N-2}{2}}$.

Let $\gamma = \sqrt{\bar{t}} + \sqrt{\bar{t}-t}, \gamma' = \sqrt{\bar{t}} - \sqrt{\bar{t}-t}$; Terracini [3] proved that S_t is achieved by the function $V(x) = (4N(\bar{t} - t)/(N-2))^{\frac{N-2}{4}}\left(|x|^{\frac{\gamma'}{\sqrt{\bar{t}}}} + |x|^{\frac{\gamma}{\sqrt{\bar{t}}}}\right)^{-\sqrt{\bar{t}}}$. Let $\epsilon > 0$, $V_\epsilon(x) = \epsilon^{-\frac{N-2}{2}}V\left(\frac{x}{\epsilon}\right)$ satisfies

$$-\Delta u = |u|^{2^*-2}u + t\frac{u}{|x|^2} \quad \text{in } R^N \setminus \{0\}; \quad u \longrightarrow 0 \text{ as } |x| \longrightarrow \infty, \tag{1.3}$$

and all the positive solutions of problem (1.3) have the form of U_ϵ. Moreover, U_ϵ achieves S_t. Let $S^t_{\alpha,\beta} = \inf_{u,v\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega\left(|\nabla u|^2+|\nabla v|^2 - t\frac{|u|^2}{|x|^2} - t\frac{|v|^2}{|x|^2}\right)dx}{\left(\int_\Omega |u|^\alpha|v|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}$. Similar to the proof of Theorem 5 in [1], we have for $t \in [0, \bar{t})$

$$S^t_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right)S_t. \tag{1.4}$$

In recent years, much attention has been paid to the existence of nonconstant solutions for the scalar semilinear elliptic problem. For example, Comte and Knaap [4] considered the following problem

$$-\Delta u - t\frac{u}{|x|^2} = |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega; \ \partial_\nu u = 0 \text{ on } \partial\Omega. \tag{1.5}$$

They proved that if $N = 3$, there exists a nontrivial solution of (1.5) with $t = 0$ for every $\lambda \in \mathbb{R}^+ \setminus \sigma(-\Delta)$, where $\sigma(-\Delta)$ is the set of eigenvalues of $-\Delta$ in $\left\{u \in H^1(\Omega) \,\Big|\, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0\right\}$; if $N \geq 4$, there exists a nontrivial solution of (1.5) with $t = 0$ for every $\lambda \geq 0$. For some $t > 0$, Han and Liu [5] considered the multiplicity of solutions of (1.5) with $\lambda < 0$. For relevant papers on problem (1.5) for $t \geq 0$, see [6–11] and the references cited therein. Note that the classical Hardy inequality does not hold any more in $H^1(\Omega)$. However, we still have the following inequalities (see [12,9,5]): let $0 < t < \bar{t} = \left(\frac{N-2}{2}\right)^2$, $N \geq 3$. Then for every $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that for any $u \in H^1(\Omega)$

$$\int_\Omega \frac{|u|^2}{|x|^2}dx \leq \frac{1+\epsilon}{\bar{t}}\int_\Omega |\nabla u|^2 dx + C(\epsilon)\int_\Omega |u|^2 dx, \tag{1.6}$$

and

$$2^{-\frac{2}{N}}S_t\left(\int_\Omega |u|^{2^*}\right)^{\frac{2}{2^*}} \leq (1+\epsilon)\int_\Omega\left(|\nabla u|^2 - t\frac{|u|^2}{|x|^2}\right)dx + C(\epsilon)\int_\Omega |u|^2 dx. \tag{1.7}$$

Problem (1.1) with $t = 0$ is motivated by [1], where Alves et al. [1] considered a class of elliptic systems with zero boundary conditions, and generalized the corresponding results in [2]. Subsequently, for the Dirichlet problem of (1.1) with $t = 0$, Han [13,14] proved the multiplicity of solutions, Liu and Han [15] established infinitely many solutions in higher values of dimension $N \geq 7$. For relevant papers, refer to [16–20] and the references cited therein.

Devillanova and Solimini [21] have recently considered $-\Delta u = |u|^{2^*-2}u + \lambda u$ for $u \in H_0^1(\Omega)$, and for $N \geq 7$, they obtained infinitely many solutions for every $\lambda > 0$. Two important methods employed by Devillanova and Solimini are concentration estimates and the lower bound of the augmented Morse index on min-max points (see [22]), which seems to be not applicable to the case of (1.1) directly. Since for $t \in (0, \bar{t})$, positive solutions of problem (1.1) may have singularity at the origin (see [5,23]), we cannot establish the uniform bound through concentration estimates and the lower bound of the augmented Morse index for solutions of (1.1) as in [21], which are main difficulties in the establishment of infinitely many energy-positive solutions of (1.1).

Let X be a Banach space. The functional $J \in C^1(X, \mathbb{R})$ is said to satisfy the $(P.S.)_c$ condition if any sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ strongly in X^* as $n \longrightarrow \infty$ contains a subsequence converging in X to a critical point of J. Assume that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $H^1(\Omega) \times H^1(\Omega)$, we are not sure whether there is a subsequence $\{(u_{n_k}, v_{n_k})\}$ such that $\lim_{k\to\infty}\int_\Omega |u_{n_k}|^\alpha|v_{n_k}|^\beta dx = \int_\Omega |u|^\alpha|v|^\beta dx$, which leads to that $I_{\lambda,\mu}$ does not satisfy the $(P.S.)_c$ condition for any $c > 0$.