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Sandwich systems

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ABSTRACT

The variational approach for solving nonlinear problems eventually leads to the search for critical points of related functionals. In case of semibounded functionals, one can look for extrema. Otherwise, one is forced to use other methods. There are several approaches. In this paper we apply a new approach which is successful in solving a large class of problems. Applications are given.

1. Introduction

Many problems arising in science and engineering call for the solving of the Euler equations of functionals, i.e., equations of the form

$$G'(u) = 0,$$

where $G(u)$ is a C^1 functional (usually representing the energy) arising from the given data. The classical approach was to look for maxima or minima. If the functional is bounded from below, one can show that there is a Palais–Smale (PS) sequence satisfying

$$G(u_k) \to a, \quad G'(u_k) \to 0 \tag{1.1}$$

for $a = \inf G$. If the sequence has a convergent subsequence, this will produce a minimum.

Actually, one can do better. If the functional $G(u)$ is bounded from below, then there exists a sequence satisfying

$$G(u_k) \to a, \quad (1 + \|u_k\|)G'(u_k) \to 0 \tag{1.2}$$

for $a = \inf G$. Such a sequence is called a Cerami sequence. It will have a convergent subsequence even in cases when it is unknown whether or not the corresponding PS sequence has one.

However, when the functional is not semibounded, the matter is more complicated and can have different features; thus, different devices must be used according to the difficulties one has to face (mountain pass theorem, saddle point theorem, linking, restriction to the Nehari manifold, Benci–Rabinowitz theorem, etc.). Here we present an approach which produces sequences similar to (1.2) for functionals which are not semibounded. They are not quite Cerami sequences, but they are just as effective in applications.

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As an illustration, consider the following sandwich theorem.

Theorem 1. *Let N be a closed subspace of a Hilbert space E and let $M = N^{\perp}$. Assume that at least one of the subspaces M, N is finite dimensional. Let G be a C^1-functional on E such that*

$$m_0 := \inf_{w \in M} G(w) \neq -\infty \tag{1.3}$$

and

$$m_1 := \sup_{v \in N} G(v) \neq \infty. \tag{1.4}$$

Then for every sequence $\nu_k \to \infty$ there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \quad m_0 \leq c \leq m_1, \quad (\nu_k + \|u_k\|)\|G'(u_k)\| \leq 1. \tag{1.5}$$

The advantage of obtaining such a sequence is that the additional structure allows one to prove the convergence of a subsequence in cases where a corresponding PS sequence need not have a converging subsequence. We shall illustrate this point in Section 4.

We search for sets A, B such that

$$a_0 := \sup_A G < \infty, \qquad b_0 := \inf_B G > -\infty \tag{1.6}$$

implies, for each sequence $\nu_k \to \infty$, the existence of a sequence $\{u_k\}$ satisfying

$$G(u_k) \to c, \quad b_0 \leq c \leq a_0, \quad (\nu_k + \|u_k\|)\|G'(u_k)\| \leq C. \tag{1.7}$$

In this case we say that the sets A, B form a strong sandwich pair. Our method of finding such pairs centers about the construction of a collection $\mathcal{K}$ of subsets K such that

$$A \in \mathcal{K}, \qquad B \cap K \neq \phi, \quad K \in \mathcal{K} \tag{1.8}$$

implies that A, B forms a strong sandwich pair. This approach differs from our previous work in [1] which relied on linking.

Our main theorems are presented in Sections 2 and 3. Proofs are given in Section 5 and applications are given in Section 4.

2. Flows

Let Q be a subset of a Banach space E, and let Σ_Q be the set of all continuous maps $\sigma = \sigma(t)$ from $E \times [0, 1]$ to E such that

1. $\sigma(0)$ is the identity map,
2. for each $t \in [0, 1]$, $\sigma(t)$ is a homeomorphism of E onto E,
3. $\sigma'(t)$ is piecewise continuous and satisfies

$$\|\sigma'(t)u\| \leq C(1 + d(\sigma(t)u, Q)), \quad u \in E. \tag{2.1}$$

The mappings in Σ_Q are called *flows*. We note the following.

Remark 2. If σ_1, σ_2 are in Σ_Q, define $\sigma_3 = \sigma_1 \circ \sigma_2$ by

$$\sigma_3(s) = \begin{cases} \sigma_1(2s), & 0 \leq s \leq \dfrac{1}{2}, \\ \sigma_2(2s-1)\sigma_1(1), & \dfrac{1}{2} < s \leq 1. \end{cases}$$

Then $\sigma_1 \circ \sigma_2 \in \Sigma_Q$.

Proof. The first two properties are obvious. To check the third, note that

$$\sigma_3'(s) = \begin{cases} 2\sigma_1'(2s), & 0 \leq s \leq \left(\dfrac{1}{2}\right)_-, \\ 2\sigma_2'(2s-1)\sigma_1(1), & \left(\dfrac{1}{2}\right)_+ \leq s \leq 1. \end{cases}$$

Thus, if

$$\|\sigma_i'(t)u\| \leq C_i(1 + d(\sigma_i(t)u, Q)), \quad u \in E, i = 1, 2, \tag{2.2}$$