J. Math. Anal. Appl. 399 (2013) 100–107

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Directional short-time Fourier transform

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ARTICLE INFO

Article history:
Received 3 September 2011
Available online 6 October 2012
Submitted by Pierre Lemarie-Rieusset

Keywords:
Short-time Fourier transform
Ridgelet
Continuous ridgelet transform
Gabor ridge function
Directional short-time Fourier transform
Inversion formula

ABSTRACT

A directionally sensitive variant of the short-time Fourier transform is introduced which sends functions on $\mathbb{R}^n$ to those on the parameter space $S^{n-1} \times \mathbb{R} \times \mathbb{R}^n$. This transform, which is named *directional short-time Fourier transform* (DSTFT), uses functions in $L^\infty(\mathbb{R})$ as window and is related to the celebrated Radon transform. We establish an orthogonality relation for the DSTFT and explore some operator-theoretic aspects of the transform, mostly in terms of proving a variant of the Hausdorff–Young inequality. The paper is concluded by some reconstruction formulas.

1. Introduction and preliminaries

The continuous wavelet and short-time Fourier transforms are important mathematical tools in areas like signal and image processing. Almost one decade after the first appearance of wavelets, E.J. Candés introduced a directionally sensitive variant of continuous wavelet transforms, known as the continuous ridgelet transform, [1–3]. Candés' idea was to construct from a Schwarz function $\psi : \mathbb{R} \to \mathbb{R}$ a family of building blocks

$$\psi_{a,b,\xi}(x) := a^{-1/2}\,\psi\left(\frac{\xi \cdot x - b}{a}\right); \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $a \in (0, \infty)$, $b \in \mathbb{R}$ and $\xi \in S^{n-1}$, and then use them to define the continuous ridgelet transform of appropriate $f : \mathbb{R}^n \to \mathbb{C}$ as a function on $(0, \infty) \times \mathbb{R} \times S^{n-1}$ by

$$R_\psi f(a, b, \xi) = \langle f, \psi_{a,b,\xi} \rangle.$$

According to (1.1), the function $\psi_{a,b,\xi}$ behaves like a one-dimensional wavelet in the direction of ξ and is constant in its orthogonal complement.

The short-time Fourier transform (STFT) of $f \in L^2(\mathbb{R}^n)$ with respect to a window $g \in L^2(\mathbb{R}^n)$, [4,5], is the function defined on $\mathbb{R}^n \times \mathbb{R}^n$ via

$$V_g f(x, w) = \int_{\mathbb{R}^n} f(t)\overline{g(t - x)}e^{-2\pi i w\cdot t}dt. \tag{1.2}$$

Assuming that T_x and M_w are the operators of translation and modulation on $L^2(\mathbb{R}^n)$ given by

$$(T_x g)(y) = g(y - x), \qquad (M_w g)(y) = e^{2\pi i w\cdot y}g(y), \tag{1.3}$$

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doi:10.1016/j.jmaa.2012.09.053

we may interpret the STFT in each of the forms

$$V_g f(x, w) = \langle f, M_w T_x g\rangle \tag{1.4}$$

$$= \widehat{f \cdot T_x \overline{g}}(w), \tag{1.5}$$

where the last function is the Fourier transform of $f \cdot T_x\overline{g}$.

There is a transform, intimately related to the STFT, that is obtained from changing the order of translation and modulation operators in (1.4). As a matter of notation, let us denote this transform by U_g:

$$U_g f(x, w) = \langle f, T_x M_w g\rangle = e^{2\pi i x\cdot w}\, V_g f(x, w). \tag{1.6}$$

In [6], Grafakos and Sansing introduced a directionally sensitive variant of U_g, which in view of (1.6) provides such a variant of the STFT. The basic idea in [6] was to transform an appropriate function f defined on $\mathbb{R}^n$, with respect to some Schwarz function g defined on $\mathbb{R}$, to a function on $S^{n-1} \times \mathbb{R} \times \mathbb{R}$:

$$(\xi, x, w) \mapsto \int_{\mathbb{R}^n} f(t)\overline{g(\xi\cdot t - x)}e^{-2\pi i w(\xi\cdot t - x)}dt. \tag{1.7}$$

As in the case of ridgelets, a so-called "Gabor ridge function"

$$t \mapsto g(\xi\cdot t - x)e^{2\pi i w(\xi\cdot t - x)}$$

behaves like a one-dimensional "Gabor function" in the direction of ξ and is constant in its orthogonal complement. (It should be mentioned that the transform sending f to the function given by (1.7) does not provide a full reconstruction of f, and this led the above mentioned authors to use "weighted Gabor ridge functions" for the analysis and synthesis of the functions. See [6] for details.) The directionally sensitive variant of the STFT mentioned above can be obtained from (1.7) by pulling $e^{2\pi i x w}$ out of the integral, i.e., it is the transform that sends f to a function defined on $S^{n-1} \times \mathbb{R} \times \mathbb{R}$ by

$$(\xi, x, w) \mapsto \int_{\mathbb{R}^n} f(t)\overline{g(\xi\cdot t - x)}e^{-2\pi i w(\xi\cdot t)}dt. \tag{1.8}$$

Noticing (1.8), we find that this transform is a directionally sensitive variant of the STFT that corresponds to its interpretation (1.4); if we interpret the STFT as in (1.4), we have to enter the directional parameter ξ in both the window g and the exponential function defining the modulation operator.

The aim of the present paper is to introduce a directionally sensitive variant of the STFT that is based on the interpretation (1.5). In fact, if we think of the STFT as in (1.5), we are led to enter the directional parameter ξ only in the window g. This is because of the scope of the STFT, best represented in (1.5): to localize the Fourier transform by multiplying the function to be analyzed by translations of an appropriate window and then taking Fourier transform.

In summary, we are interested in an integral transform that sends a function f defined on $\mathbb{R}^n$ to one on $S^{n-1} \times \mathbb{R} \times \mathbb{R}^n$ given by

$$\mathcal{DS}_g f(\xi, x, w) := \int_{\mathbb{R}^n} f(t)\overline{g(\xi\cdot t - x)}e^{-2\pi i t\cdot w}dt, \tag{1.9}$$

where g is an appropriate window defined on $\mathbb{R}$. It follows from (1.9) that, under certain assumptions on f and g, $f \mapsto \mathcal{DS}_g f$ can still be a certain Fourier transform, and we can therefore legitimately name it the *directional short-time Fourier transform* (DSTFT). Here, the parameters ξ and x appear in the "directional window"

$$g_{\xi,x} : t \in \mathbb{R}^n \mapsto g(\xi\cdot t - x), \tag{1.10}$$

which behaves like a one-dimensional window in the direction of ξ and is constant in its orthogonal complement, and as in (1.2), the parameter w is used only for taking the Fourier transform.

To introduce the DSTFT more completely, we organized this paper as follows. In the remainder of the present section, we fix our notation, make our conventions and present the required preliminaries. In Section 2, the DSTFT is formally introduced and conditions on f and g are imposed to ensure that the integral in (1.9) is absolutely convergent. Relation to the Radon transform and an orthogonality relation are also established in this section. The section is concluded by proving a Hausdorff–Young inequality for the DSTFT and one interesting corollary of it. The inequality allows better understanding of the operator theoretic aspects of the transform. Finally, in Section 3, weak-sense and pointwise inversion formulas and an approximate reconstruction procedure are obtained for the DSTFT.

Suppose f and h are measurable functions defined on the same measure space (X, N, ν). If $\int_X f(x)\overline{h(x)}d\nu(x)$ is absolutely convergent, we denote it by $\langle f, h\rangle$.

For an arbitrary function g defined on $\mathbb{R}^m$ and $x, w \in \mathbb{R}^m$, $m \geq 1$, we define the functions $T_x g$ and $M_w g$ on $\mathbb{R}^m$ as in (1.3). The resulting operators T_x and M_w are called translation and modulation operators, respectively. If $g \in L^1(\mathbb{R}^m)$, the Fourier transform of g will be denoted by $\widehat{g}$.

In the remainder of this paper, n is a fixed integer greater than 1. If $g : \mathbb{R} \to \mathbb{C}$ is any function and $(\xi, x) \in S^{n-1} \times \mathbb{R}$, we define $g_{\xi,x} : \mathbb{R}^n \to \mathbb{C}$ by (1.10). We also set $g_{\xi,x,w} = M_w g_{\xi,x}$ for every $w \in \mathbb{R}^n$.