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Global dynamics of a state-dependent delay model with unimodal feedback

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ABSTRACT

We obtain the global attractivity of nonnegative stationary states for a parameterized state-dependent delay equation with unimodal feedback. Under certain mild conditions, we show that the equation with unimodal type nonlinearity can generate rich dynamics as the parameter varies. To be specific, global attractivity of the positive stationary state is obtained in a set of nonnegative bounded continuous functions, when the stationary state is less than a value implicitly determined by a condition on the unimodal feedback. The general results of global attractivity are illustrated through two examples arising from population dynamics. Moreover, Hopf bifurcations are demonstrated in the examples when the positive stationary states lose global attractivity.

1. Introduction

We consider the following differential system with state-dependent delay

$$\begin{cases} \dot{x}(t) = -\mu x(t) + b(x(t-\tau(t))), & \text{(a)} \\ \dot{\tau}(t) = h(x(t), \tau(t)), & \text{(b)} \end{cases} \quad (1.1)$$

where $\mu > 0$, $x \in \mathbb{R}$, $h : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, and $b : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function which is unimodal on $[0, +\infty)$ with the following properties:

(H1) $b(0) = 0$ and there exists $M_0 > 0$ or $M_0 = +\infty$ such that

$$M_0 = \inf\{\zeta : b(\xi) > 0 \text{ for every } \xi \in (0, \zeta)\};$$

(H2) There exists a unique $\xi_0 \in (0, M_0)$ such that

$$\begin{cases} b'(\xi) > 0 & \text{if } 0 \le \xi < \xi_0, \\ b'(\xi) < 0 & \text{if } \xi > \xi_0; \end{cases} \quad (1.2)$$

(H3) $b''(\xi) < 0$ if $\xi \in (0, \xi_0)$.

Unimodal functions satisfying conditions (H1)–(H3) include the Ricker type function (see, e.g., [1]) $b(\xi) = p\xi e^{-q\xi}$ with $p > 0$, $q > 0$ and the logistic function $b(\xi) = \nu\xi(1 - \xi/K)$ with $\nu > 0$, $K > 0$. We discuss system (1.1) with these two types of functions in Section 4.

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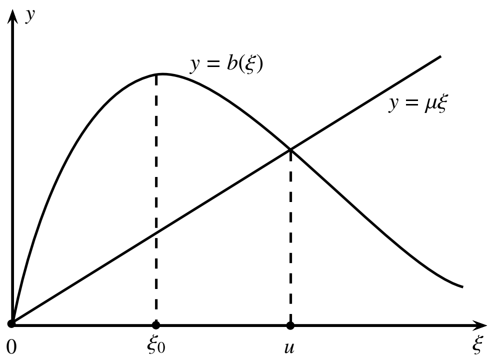

Fig. 1. Nonnegative stationary states of x in system (1.1). As $\mu \in (0, +\infty)$ varies, the nonnegative stationary states of x are the values of $\xi \geq 0$ such that $\mu\xi = b(\xi)$.

System (1.1) with the right hand side of the first equation being the form of $f(x(t - \tau(t)))$ was investigated in [2] for the existence of periodic solutions. State-dependent delay governed by a differential equation was employed in [3] modeling fish dynamics, in [4] on internet congestion control problems modeling data package queueing delay and in [5] on stabilization of turning processes. If we assume that the delay is a constant in system (1.1), then it becomes a well-known model in population dynamics with birth rate depending on a constantly delayed state variable (see, e.g., [6,7] and references therein). In principle, the assumption that the time delay is a constant in the model is justified on the basis of historical data. If the assumption that the delay is constant is not valid, an appropriate equation modeling the delay itself becomes necessary. Besides the model for a delay in the form of ordinary differential equation, other possible models include a delay of threshold type $\int_{t-\tau(t)}^{t} K(x(s))ds = c$ (see, e.g., [8]), where c is a constant and K is a given function, and a delay given by an algebraic equation $\tau(t) = r_0(x(t))$ where r_0 is a function of x (see, e.g., [9–12]).

In order to address the global dynamics, we consider differential equations with state-dependent delay governed by an ordinary differential equation, for which the Hopf bifurcation theory has been developed in [9,13]. Our research in this paper is a continuation of the work in [13] for models with state-dependent delay governed by an ordinary differential equation.

The purpose of this paper is to study the asymptotic behavior of the dynamics generated by system (1.1). More precisely, we determine the invariant closed order intervals that attract every non-trivial positive trajectory of the system, the global attractivity of the nonnegative stationary states (see Fig. 1), and the local Hopf bifurcation from the nonnegative stationary states as the system parameter varies. It is well-known that differential equations with state-dependent delay in general have no linearization in the classical sense. Therefore, many methods related to linearization for analysis of dynamics become inapplicable for such systems. See, among many others, [14,5,11,12] for investigations of local stabilities of equations with state-dependent delays. Naturally, the global dynamics for differential systems with state-dependent delay is a very challenging problem. As a starting point, we choose to study the model system (1.1) in this paper. In the settings of constant delay, global attractivity of positive stationary states of differential equations with unimodal feedback was investigated in [7,15]. Some of the techniques employed in [7] are verified to be applicable in the scenario of state-dependent delay (See Lemma 3.6). In addition to the main difference between the nature of the delays, the unimodal feedback b is allowed to be negative on $\mathbb{R}^+$ while nonnegativity was assumed in [7,15]. The global attractivity of the stationary state located in the part of domain of b where b is decreasing was not obtained in [7,15] but is investigated in the current work. Moreover, we show through two examples that Hopf bifurcation occurs when such a stationary state loses global attractivity.

We organize the paper as follows. In Section 2, we first establish the existence and non-negativity of the global solutions of system (1.1). Then we obtain in Section 3 the invariant and attractive intervals of system (1.1) with unimodal feedback b and the global attractivity of the nonnegative stationary state located in the part of the domain of b where b is nondecreasing. By using the method of fluctuations (see, e.g., [16,17]), we also establish the global attractivity of the nonnegative stationary state located at the domain of b where b is decreasing. In Section 4, we discuss the global dynamics and Hopf bifurcation for two examples.

2. Global existence and nonnegativity of solutions

For the sake of convenience, we state Theorems 1 and 2 of Driver [18] in the following lemma, which has been adapted to the context of our discussion.

Lemma 2.1 ([18]). *Consider the system*

$$\begin{cases} \dot{y}(t) = f(y(t), y(t - z(t))), \\ \dot{z}(t) = g(y(t), y(t - z(t)), z(t)) \end{cases} \tag{2.1}$$