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Global stability of travelling fronts for a damped wave equation

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ABSTRACT

The paper is concerned with the long-time behaviour of the solutions of the damped wave equation $\alpha u_{tt} + u_t = u_{xx} - V'(u)$ on $\mathbb{R}$. This equation has travelling front solutions of the form $u(x, t) = h(x-st)$. Gallay and Joly have proved in Gallay and Joly (2009) [7] that when the nonlinearity $V(u)$ is of bistable type, if the initial data are sufficiently close to the profile of a front for large $|x|$, the solution of the damped wave equation converges uniformly on $\mathbb{R}$ to a travelling front as $t \to +\infty$. In this paper, we establish a global stability result under more general assumptions on the function V, which include in particular nonlinearities of combustion type. We impose, however, more restrictive conditions on the initial data in the region $x \gg 1$. We also apply our method to the case of a monostable pushed front.

1. Introduction

This paper is concerned with the asymptotic behaviour as $t \to \infty$ of solutions of the following semilinear damped hyperbolic equation

$$\alpha u_{tt} + u_t = u_{xx} - V'(u) \tag{1.1}$$

where $\alpha > 0$ is a parameter, $V : \mathbb{R} \to \mathbb{R}$ is a smooth potential and the unknown $u = u(x, t)$ is a real-valued function of $x \in \mathbb{R}$ and $t \geq 0$.

As was observed by many authors before, the solutions of such kind of hyperbolic equations behave similarly to those of the parabolic equation $u_t = u_{xx} - V'(u)$. In the parabolic case, global stability results have been established for travelling waves using a priori estimates and comparison theorems based on the parabolic maximum principle, under various assumptions on the nonlinearity, such as the *bistable case* [1,2], *combustion case* [3,4] and the *monostable case* [5]. However, for the damped hyperbolic equation (1.1) no maximum principle is available in general, thus the proofs of all these results cannot be extended to it.

Recently, a different approach has been developed by Risler [6] to prove the global stability of bistable travelling waves. This new method is of variational nature and is therefore restricted to systems which admit a gradient structure, but it does not make any use of the maximum principle and is therefore applicable to a wide class of problems. Gallay and Joly [7] adapted it to the damped hyperbolic equation (1.1) for bistable nonlinearity. In this paper we shall extend the results of [7] to a more general setting including the combustion nonlinearity and a case of monostable nonlinearity, but with more restricted initial data. For the sake of clarity, we first concentrate on the combustion case, and state the main results in that framework. In the last section of the paper, we shall briefly indicate how the method can be adapted to obtain a partial result in the case of a monostable nonlinearity.

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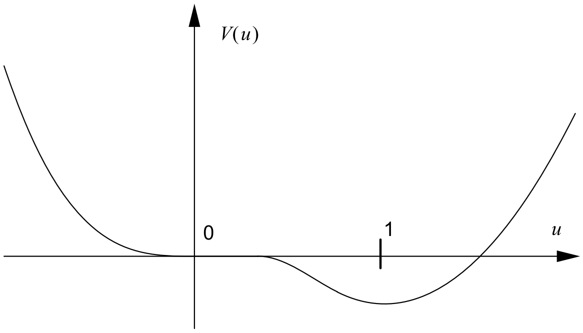

Fig. 1. A simple example of V satisfying assumptions (1.2)–(1.4).

From now on, we suppose that $V \in \mathcal{C}^3(\mathbb{R})$, and that there exist positive constants a, b such that $uV'(u) \geq au^2 - b$, for all $u \in \mathbb{R}$. We also assume

$$V(0) = 0, \qquad V'(0) = 0, \quad \text{and} \quad V''(u) \geq 0 \quad \text{near } u = 0, \tag{1.2}$$

$$V(1) < 0, \qquad V'(1) = 0, \qquad V''(1) > 0. \tag{1.3}$$

Finally we suppose that, except for $V(0)$ and $V(1)$, all critical values of V are positive:

$$\{u \in \mathbb{R} | V'(u) = 0, V(u) \leq 0\} = \{0; 1\}. \tag{1.4}$$

Assumptions (1.2)–(1.4) imply that V reaches its global minimum at $u = 1$ and has another (possibly degenerate) local minimum at $u = 0$, (see Fig. 1). Unlike in [7], we do not assume that $V''(0) > 0$, because we want to include the case of a combustion nonlinearity.

It is well-known that (see [8]), under assumptions (1.2)–(1.4), the parabolic equation $u_t = u_{xx} - V'(u)$ has a family of travelling fronts of the form $u(x, t) = h(x - c_* t - x_0)$ connecting the equilibria $u = 1$ and $u = 0$. More precisely, there exists a unique speed $c_* > 0$ such that the boundary value problem

$$\begin{cases} h''(y) + c_* h'(y) - V'(h(y)) = 0, & y \in \mathbb{R}, \\ h(-\infty) = 1, \qquad h(+\infty) = 0, \end{cases} \tag{1.5}$$

has a solution $h : \mathbb{R} \to (0, 1)$, in which case the profile h itself is unique up to a translation. Moreover $h \in \mathcal{C}^4(\mathbb{R})$, $h'(y) < 0$ for all $y \in \mathbb{R}$, and $h(y)$ converges exponentially toward its limits as $y \to \pm\infty$.

Existence of uniformly translating front solutions to damped hyperbolic equations has been proved by Hadeler [9] in a more general context. As in [10], in our case, this is simply done by setting $u(x, t) = h(\sqrt{1 + \alpha c^2}x - ct)$ and inserting into (1.1) to obtain that for any $\alpha > 0$ the damped hyperbolic equation (1.1) has a corresponding family of travelling fronts given by

$$u(x, t) = h\left(\sqrt{1 + \alpha c_*^2}x - c_* t - x_0\right), \quad x_0 \in \mathbb{R}. \tag{1.6}$$

Note that the actual speed of these waves is $s_* = c_* / \sqrt{1 + \alpha c_*^2}$.

The main convergence result will be built on the uniformly local Lebesgue spaces which provide a natural framework for the study of partial differential equations on unbounded domains, see e.g. [11–19]. So we first recall the definitions of them.

Definition 1. The uniformly local Lebesgue space $L^2_{\text{ul}}(\mathbb{R})$ is defined as

$$L^2_{\text{ul}}(\mathbb{R}) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}) | \|u\|_{L^2_{\text{ul}}} < \infty, \ \lim_{\xi \to 0} \|T_\xi u - u\|_{L^2_{\text{ul}}} = 0 \right\}, \tag{1.7}$$

where T_ξ denotes the translation operator: $(T_\xi u)(x) = u(x - \xi)$.

Definition 2. For any $k \in \mathbb{N}$, the uniformly local Sobolev space $H^k_{\text{ul}}(\mathbb{R})$ is defined as

$$H^k_{\text{ul}}(\mathbb{R}) := \left\{ u \in H^k_{\text{loc}}(\mathbb{R}) | \partial^j u \in L^2_{\text{ul}}(\mathbb{R}) \text{ for } j = 0, 1, 2, \dots, k \right\}, \tag{1.8}$$

which is equipped with the natural norm $\|u\|_{H^k_{\text{ul}}} = \left(\sum_{j=0}^k \|\partial^j u\|^2_{L^2_{\text{ul}}} \right)^{1/2}$.