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A basic class of symmetric orthogonal polynomials of a discrete variable

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ABSTRACT

By using a generalization of Sturm–Liouville problems in discrete spaces, a basic class of symmetric orthogonal polynomials of a discrete variable with four free parameters, which generalizes all classical discrete symmetric orthogonal polynomials, is introduced. The standard properties of these polynomials, such as a second order difference equation, an explicit form for the polynomials, a three term recurrence relation and an orthogonality relation are presented. It is shown that two hypergeometric orthogonal sequences with 20 different weight functions can be extracted from this class. Moreover, moments corresponding to these weight functions can be explicitly computed. Finally, a particular example containing all classical discrete symmetric orthogonal polynomials is studied in detail.

1. Introduction

Some special functions of mathematical physics such as classical orthogonal polynomials and cylindrical functions [1], are solutions of a differential equation of hypergeometric type [2,1,3]

$$\sigma(x)y''(x)+\tau(x)y'(x)+\lambda y(x)=0, \tag{1}$$

and extendible by changing Eq. (1) to a difference equation of the form

$$\tilde{\sigma}(x(s))\frac{\Delta}{\nabla x_1(s)}\left[\frac{\nabla y(s)}{\nabla x(s)}\right]+\frac{\tilde{\tau}(x(s))}{2}\left[\frac{\Delta y(s)}{\Delta x(s)}+\frac{\nabla y(s)}{\nabla x(s)}\right]+\lambda y(s)=0, \tag{2}$$

where

$$\Delta x(s)=x(s+1)-x(s),\qquad \nabla x(s)=\Delta x(s-1),\qquad \frac{\Delta}{\Delta x(s)}f(s)=\frac{f(s+1)-f(s)}{x(s+1)-x(s)},$$

$\tilde{\sigma}(x(s))$ and $\tilde{\tau}(x(s))$ are polynomials of degree at most two and one, respectively, in $x(s)$, λ is a constant, and $x_1(s)=x(s+1/2)$.

The difference equation (2), which is obtained by approximating the differential equation (1) on a non-uniform lattice, is of much importance [3] as its particular solutions have been applied in quantum mechanics, theory of group representations

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and especially computational mathematics, where one can point to the Clebsch–Gordan and Racah coefficients with wide applications in atomic and nuclear spectroscopy. There exist different approaches for the analysis of orthogonal polynomials of a discrete variable running from the classical Refs. [4,5] to the recent monograph [6], which is a basic reference on orthogonal polynomials.

Also there exists a number of numerical and symbolic methods for solving hypergeometric equations of type (1) or (2), which are of interest in applications, particularly for cases containing symmetric solutions, such as resolution of the Gibbs phenomenon [7,8], Fourier–Kravchuk transform used in Optics [9], approximation of harmonic oscillator wave functions [10], tissue segmentation of human brain MRI through preprocessing [11], reconstructions for electromagnetic waves in the presence of a metal nanoparticle [12], efficient determination of the critical parameters and the statistical quantities for Klein–Gordon and sine–Gordon equations with a singular potential [13], image representation [14,15] and quantitative theory for the lateral momentum distribution after strong-field ionization [16].

The main aim of this paper is to introduce a basic class of symmetric orthogonal polynomials of a discrete variable with four free parameters, which is the polynomial solution of a symmetric generalization of Eq. (2) on the uniform lattice $x(s) = s$. Computational aspects of these new polynomials are described in detail giving their explicit representation as well as the three-term recurrence relation they satisfy. A full classification of weight functions and orthogonality supports is given together with computing the moments of the aforesaid weights. From this class all classical symmetric orthogonal polynomials of a discrete variable can be recovered (Section 6), and its limit relation with the continuous type of generalized classical symmetric orthogonal polynomials is given (see Remark 2).

A regular Sturm–Liouville problem of continuous type is a boundary value problem of the form

$$\frac{d}{dx}\left(k(x)\frac{dy_n(x)}{dx}\right) + (\lambda_n \varrho(x) - q(x))\, y_n(x) = 0 \quad (k(x) > 0, \varrho(x) > 0), \tag{3}$$

which is defined on an open interval (a, b), and has the boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = 0, \qquad \alpha_2 y(b) + \beta_2 y'(b) = 0, \tag{4}$$

where α_1, α_2 and β_1, β_2, are given constants and $k(x)$, $k'(x)$, $q(x)$, and $\varrho(x)$ in (3) are to be assumed continuous for $x \in [a, b]$. If one of the boundary points a and b is singular (i.e. $k(a) = 0$ or $k(b) = 0$), the problem is transformed to a singular Sturm–Liouville problem.

Let y_n and y_m be two eigenfunctions of the operator $D(k(x)D) - q(x)I$, where D is the standard derivative operator. According to Sturm–Liouville theory [1], they are orthogonal with respect to the weight function $\varrho(x)$ under the given conditions (4) and satisfy the orthogonality relation

$$\int_a^b \varrho(x) y_n(x) y_m(x) dx = \left(\int_a^b \varrho(x) y_n^2(x) dx\right) \delta_{n,m}.$$

Many of special functions are orthogonal solutions of a regular or singular Sturm–Liouville problem having the symmetry property ($\phi_n(-x) = (-1)^n \phi_n(x)$) so that have found valuable applications in physics and engineering, as already mentioned. In [17], the classical equation (3) is symmetrically extended as follows.

Theorem 1 *([17]). Let $\phi_n(-x) = (-1)^n \phi_n(x)$ be a sequence of symmetric functions satisfying the equation*

$$A(x)\phi_n''(x) + B(x)\phi_n'(x) + (\lambda_n C(x) + D(x) + \sigma_n E(x))\, \phi_n(x) = 0, \tag{5}$$

where

$$\sigma_n = \frac{1-(-1)^n}{2} = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases} \tag{6}$$

and λ_n is a sequence of constants. If $A(x)$, $(C(x) > 0)$, $D(x)$ and $E(x)$ are even functions and $B(x)$ is odd then

$$\int_{-\nu}^{\nu} \varrho^*(x)\phi_n(x)\phi_m(x) dx = \left(\int_{-\nu}^{\nu} \varrho^*(x)\phi_n^2(x) dx\right) \delta_{n,m},$$

where

$$\varrho^*(x) = C(x)\exp\left(\int \frac{B(x) - A'(x)}{A(x)} dx\right) = \frac{C(x)}{A(x)}\exp\left(\int \frac{B(x)}{A(x)} dx\right). \tag{7}$$

The weight function defined in (7) must be positive and even on $[-\nu, \nu]$ and the function

$$A(x)K(x) = A(x)\exp\left(\int \frac{B(x) - A'(x)}{A(x)} dx\right) = \exp\left(\int \frac{B(x)}{A(x)} dx\right)$$

must vanish at $x = \nu$, i.e. $A(\nu)K(\nu) = 0$. In this way, since $K(x) = \varrho^*(x)/C(x)$ is an even function so $A(-\nu)K(-\nu) = 0$ automatically.