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A class of dissipative wave equations with time-dependent speed and damping

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ABSTRACT

We study the long time behavior of the energy for wave-type equations with time-dependent speed and damping:

$$u_{tt} - \lambda(t)^2 \Delta u + b(t) u_t = 0.$$

We investigate the interaction between the speed of propagation $\lambda(t)$ and the damping coefficient $b(t)$, showing how to describe the dissipative effect on the energy. We study a class of dissipations for which the equation keeps its hyperbolic structure and properties.

1. Introduction

Let us consider in $[0, \infty) \times \mathbb{R}^n$, with space dimension $n \geq 1$, the Cauchy problem

$$\begin{cases} u_{tt} - \lambda(t)^2 \Delta u + b(t) u_t + \lambda(t) \tilde{b}(t) \cdot \nabla u + e(t) u = 0, \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \tag{1}$$

where by $\tilde{b}(t) = (b_j(t))_{j=1,\dots,n}$ we denote the vector with components $b_j(t)$, that is,

$$\tilde{b}(t) \cdot \nabla u = \sum_{j=1}^{n} b_j(t)\, u_{x_j}.$$

It is well known that if the coefficients are sufficiently regular and the equation is strictly hyperbolic, that is, $\lambda(t) > 0$, then the Cauchy problem (1) is globally well-posed in $\mathcal{C}^\infty$ and in all Sobolev spaces with no loss of regularity. However, if we consider the energy of the solution to (1) given by

$$E_\lambda(t) = \|u_t(t, \cdot)\|_{L^2}^2 + \lambda(t)^2 \|\nabla u(t, \cdot)\|_{L^2}^2, \tag{2}$$

then we can observe many different effects for the behavior of $E(t)$ as $t \to \infty$, according to the properties of the speed of propagation $\lambda(t)$ and of the other coefficients of the equation.

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We first consider the Cauchy problem for the homogeneous equation:

$$u_{tt} - \lambda(t)^2 \Delta u = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x). \tag{3}$$

If $0 < \lambda_0 \le \lambda(t) \le \lambda_1$ for some $\lambda_0, \lambda_1 > 0$ then the energy $E_\lambda(t)$ is equivalent to

$$E_1(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2, \tag{4}$$

but the oscillations of $\lambda = \lambda(t)$ may have a deteriorating influence [1] on the energy behavior for the solution to (3). On the other hand, if $\lambda \in \mathcal{C}^2$ and

$$|\lambda^{(k)}(t)| \le C_k (1+t)^{-k}, \quad \text{for } k = 1, 2,$$

then the so-called *generalized energy conservation* property holds [2], that is,

$$C_0 E_1(0) \le E_1(t) \le C_1 E_1(0). \tag{5}$$

If $\lambda(t) \ge \lambda_0 > 0$ and $\lambda(t) \to \infty$ as $t \to \infty$ in (3) then one can prove the estimate

$$C_0(u_0, u_1)\lambda(t) \le E_\lambda(t) \le C_1 \lambda(t)\, E(0), \quad \text{where } E(0) := \left(\|u_0\|_{H^1} + \|u_1\|_{L^2}\right), \tag{6}$$

for the solution to (3), by assuming sufficient regularity for $\lambda(t)$ and some kind of control on its oscillations [3]. Referred to this energy, an increasing speed of propagation can be considered as a dissipative effect (since $\|\nabla u(t,\cdot)\| \le C_1 \lambda(t)^{-1} E(0)$). A fundamental difference with (5) is that in the right-hand side term of (6) it appears the H^1 norm of u_0, not only the L^2 norm of its gradient.

We address the interested reader to [4–7] for other results concerning (3).

Let us consider the wave equation with time-dependent damping term $b(t)u_t$, with $b(t) > 0$:

$$u_{tt} - \Delta u + b(t) u_t = 0. \tag{7}$$

The dissipation produced by $b(t)u_t$ may be classified [8] as *non effective* if the Eq. (7) has the same asymptotic properties of the free wave equation, *effective* if the equation inherits some properties related to the parabolic equation $b(t)u_t - \Delta u = 0$. In particular, if $tb(t) < 1$ for large times [9] or in the special case $b(t) = \mu(1+t)^{-1}$ for $\mu \in (0, 2]$ (see [10]), the following estimate holds for Eq. (7):

$$E_1(t) \le C\gamma(t)\, E(0), \quad \text{where } \gamma(t) := \exp\left(-\int_0^t b(\tau)d\tau\right). \tag{8}$$

In this case, the dissipation is *non effective* for the L^2–L^2 estimates of the energy.

We will not study *effective* dissipations in this paper, but we address the interested reader to [11–14]. Neither will we study L^p–L^q estimates, with $(p,q) \ne (2,2)$ (see, for instance, [1,2,15]).

Theorem 2 extends energy estimates (6) and (8) to a more complex situation with a unified approach. In particular, we prove the energy estimate $E_\lambda(t) \le \lambda(t)\gamma(t)E(0)$ for the solution to (1), under suitable assumptions which take into account the interaction between the speed of propagation $\lambda(t)$ and the term $b(t)u_t$. In particular, $\lambda'(t) + b(t)\lambda(t)$ is *almost-positive* (see Definition 1).

Moreover, in Theorem 2 we assume hypotheses which allow us to exclude contributions to the energy behavior coming from the other coefficients, namely $b_j(t)$ and $e(t)$. On the other hand, in Theorem 3 we also include a possible damaging contribution to the energy estimate coming from the drift terms $b_j(t)u_{x_j}$.

The class of dissipation which we study are *non effective*, in the sense that the damping term $b(t)u_t$ produces a factor $\gamma(t)$ in the L^2–L^2 estimate of the energy, with respect to the estimate (6) for (3). In [16], we show how to extend this approach to higher order equations.

2. Main results

Notation 1. Let $f, g : [0,\infty) \to (0,\infty)$ be two strictly positive functions. We use the notation $f \approx g$ if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(t) \le f(t) \le C_2 g(t)$ for all $t \ge 0$. If the inequality is one-sided, namely, if $f(t) \le Cg(t)$ (resp. $f(t) \ge Cg(t)$) for all $t \ge 0$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

In particular $f \approx 1$ means that $C_1 \le f(t) \le C_2$ for some constants C_1, C_2.

Notation 2. Through this paper, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* (resp. *strictly increasing*, *decreasing*, *strictly decreasing*) if $f(x) \le f(y)$ (resp. $f(x) < f(y)$, $f(x) \ge f(y)$, $f(x) > f(y)$) for any $x, y \in \mathbb{R}$ such that $x < y$.