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Existence and multiplicity of semiclassical solutions for asymptotically Hamiltonian elliptic systems

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ABSTRACT

This paper is concerned with the following nonperiodic Hamiltonian elliptic system

$$\begin{cases} -\varepsilon^2\triangle u + V(x)u = H_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\triangle v + V(x)v = -H_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u(x)\to 0 \text{ and } v(x)\to 0 & \text{as } |x|\to\infty, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, and the potential V is bounded below, and H is asymptotically linear in z as $|z|\to\infty$ with $z=(u,v)$. By applying a generalized linking theorem of strongly indefinite functionals, we obtain the existence of multiple semiclassical solutions for the above system.

1. Introduction and main results

In this paper we study the following nonperiodic elliptic system in Hamiltonian form:

$$(\mathcal{P}_\varepsilon)\quad \begin{cases} -\varepsilon^2\triangle u + V(x)u = H_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\triangle v + V(x)v = -H_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u(x)\to 0 \text{ and } v(x)\to 0 & \text{as } |x|\to\infty, \end{cases}$$

where $N\ge 1$, $V\in C(\mathbb{R}^N,\mathbb{R})$ and $H\in C^1(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R})$. Set $\mathcal{J}_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Thenthe system $(\mathcal{P}_\varepsilon)$ can be written as

$$-\varepsilon^2\Delta z + V(x)z = \mathcal{J}_0 H_z(x,z), \qquad z(x)=(u(x),v(x))\to 0 \quad \text{as } |x|\to\infty,$$

which can be regarded as the stationary system of the nonlinear vector Schrödinger equation

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \gamma(x)\phi - \mathcal{J}_0 f(x,|\phi|)\phi,$$

where $\phi(x,t)=z(x)e^{-\frac{iEt}{\hbar}}$, $V(x)=\gamma(x)-E$, $\varepsilon^2=\frac{\hbar^2}{2m}$ and $H_z(x,z)=f(x,|z|)z$. Set $\lambda=\varepsilon^{-2}$. Then the system $(\mathcal{P}_\varepsilon)$ is equivalent to the following one

$$(\mathcal{P}_\lambda)\quad \begin{cases} -\triangle u + \lambda V(x)u = \lambda H_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\triangle v + \lambda V(x)v = -\lambda H_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u(x)\to 0 \text{ and } v(x)\to 0 & \text{as } |x|\to\infty. \end{cases}$$

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If the parameter $\varepsilon = 1$ and Ω is a bounded domain, many mathematicians are devoted to study the existence and multiplicity of solutions for the system $(\mathcal{P}_\varepsilon)$ in recent years. For example, see [1–6] and the references therein. Benci and Rabinowitz [3] first considered the system $(\mathcal{P}_\varepsilon)$. By using the direct min–max method, they obtained solutions for the system $(\mathcal{P}_\varepsilon)$ in the space of $H_0^1(\Omega) \times H_0^1(\Omega)$. Later, Hulshof and van der Vorst [2] also considered the system $(\mathcal{P}_\varepsilon)$. Instead of working in the space of $H_0^1(\Omega) \times H_0^1(\Omega)$, the authors used a suitable family of products of fractional Sobolev spaces, and turned out that these kinds of spaces are the natural ones for this problem. By doing so, the authors also proved that the system $(\mathcal{P}_\varepsilon)$ has existence of solutions. In [1] de Figueiredo and Felmer considered the super-quadratic case for the system $(\mathcal{P}_\varepsilon)$. By applying a critical point theorem, they acquired solutions for the system $(\mathcal{P}_\varepsilon)$. In [6], Kryszewski and Szulkin considered the following system

$$(\mathcal{X})\ \begin{cases} -\triangle u = F_v(x, u, v) & \text{in } \Omega \subset \mathbb{R}^N, \\ -\triangle v = F_u(x, u, v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

By developing an infinite-dimensional Morse theory for strongly indefinite functionals, for $N \geq 1$, they obtained one weak solution for the system $(\mathcal{X})$. In particular, if $N = 1$, they proved that the system $(\mathcal{X})$ has at least two nontrivial solutions. Recently, by developing a critical point theorem for strongly indefinite functionals, de Figueiredo and Ding [4] studied the system

$$\begin{cases} -\triangle u = H_u(x, u, v) & \text{in } \Omega \subset \mathbb{R}^N, \\ -\triangle v = -H_v(x, u, v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where H satisfies $H(x, u, v) \sim |u|^p + |v|^q + R(x, u, v)$ with $\lim_{|u|+|v|\to\infty} \frac{R(x,u,v)}{|u|^p+|v|^q} = 0$, $1 < p < 2^* = \frac{2N}{N-2}$ and $q > 1$. Under some additional conditions on R, they proved the existence of multiple solutions for the above system. More recently, de Figueiredo et al. [5] have treated the following autonomous system via an Orlicz space approach

$$\begin{cases} -\triangle u = g(v)\, u(x) > 0 & \text{in } \Omega, \\ -\triangle v = f(u)\, v(x) > 0 & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Under some superlinear conditions on f and g, the authors obtained one solution for the system.

Up till now, some authors considered the system $(\mathcal{P}_\varepsilon)$ in the space of $\mathbb{R}^N$. For the parameter $\varepsilon = 1$, the papers [7–17] proved the existence of multiple nontrivial solutions for the system $(\mathcal{P}_\varepsilon)$. Most of them focused on the case $V \equiv 1$. The main difficulty of this problem is the lack of compactness for Sobolev's embedding theorem. In the early results [8–11], the authors managed to recover the compactness for Sobolev's embedding, by working in the space of radially symmetric functions. In this way, they proved the existence of radial solutions for the system $(\mathcal{P}_\varepsilon)$. Later, by using the dual variational method, the papers [12–15] also overcome the lack of compactness for Sobolev's embedding theorem. Therefore, they obtained solutions for the system $(\mathcal{P}_\varepsilon)$. Recently, in [18], Bartsch and Ding have developed a generalized linking theorem for the strongly indefinite functionals (see [19,20] for related results), which provided another way to deal with such problems. The papers [7,16,21] considered the following system

$$(\mathcal{Z})\ \begin{cases} -\triangle u + V(x)u = R_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\triangle v + V(x)v = R_u(x, u, v) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \text{ and } v(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

By applying the generalized linking theorems of [18], they proved the existence of solutions for the system $(\mathcal{Z})$. Let Ω be a domain of $\mathbb{R}^N$, not necessarily bounded, with a smooth or empty boundary. If the parameter $\varepsilon > 0$ is small enough, the paper [22] proved the existence of nonconstant positive solutions under Neumann boundary conditions. Later, under the Dirichlet conditions at the boundary, the paper [23] proved that $(\mathcal{P}_\varepsilon)$ admits classical positive solutions $u_\varepsilon, v_\varepsilon \in C^2(\Omega) \cap C^1(\bar{\Omega}) \cap H_0^1(\Omega)$ for any $\varepsilon > 0$ small enough. Moreover, $u_\varepsilon, v_\varepsilon$ concentrate at a prescribed finite number of local minimum points of $V(x)$, possibly degenerate, as $\varepsilon \to 0$. In the paper [24], the authors proved that the least energy solutions to the system $(\mathcal{P}_\varepsilon)$ concentrate, as $\varepsilon \to 0$, at a point of Ω which maximizes the distance to the boundary of Ω. For the whole space $\mathbb{R}^N$, the papers [25,13] considered the system

$$(\mathcal{N}_\varepsilon)\ \begin{cases} -\varepsilon^2\triangle u + V(x)u = R_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\triangle v + V(x)v = R_u(x, u, v) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \text{ and } v(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

In the paper [13], assume that $R(x, u, v) = h_1(x)|v|^q + h_2(x)|u|^p$, where the exponents $p, q > 1$ are below the critical hyperbola, that is, $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N} (N \geq 3)$. The authors obtained positive solutions for the system $(\mathcal{N}_\varepsilon)$. Later, assume that $R(x, u, v) = K(x)\int_0^v g(s)ds + H(x)\int_0^u f(s)ds$ and f, g have superlinear and subcritical growth at infinity, and V, H, K are positive and locally Hölder continuous. Ramos [25] proved the existence and multiplicity positive solutions of the system $(\mathcal{N}_\varepsilon)$. Nearly, the paper [26] considered nonperiodic superquadratic case for the system $(\mathcal{P}_\varepsilon)$. By applying the generalized