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# Bounded log-harmonic functions with positive real part

H.E. Özkan *, Y. Polatoğlu

*Department of Mathematics and Computer Sciences, İstanbul Kültür University, Istanbul, Turkey*

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ABSTRACT

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z \mid |z| < 1\}$, and let $B$ be the set of all functions $w(z) \in H(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation $\overline{f_{\bar{z}}} = w(z)\left(\overline{f}/f\right)f_z$, where $w(z)$ is the second dilatation of $f$ and $w(z) \in B$. In the present paper we investigate the set of all log-harmonic mappings $R$ defined on the unit disc $\mathbb{D}$ which are of the form $R = H(z)\overline{G(z)}$, where $H(z)$ and $G(z)$ are in $H(\mathbb{D})$, $H(0) = G(0) = 1$, and $Re(R) > 0$ for all $z \in \mathbb{D}$. The class of such functions is denoted by $\mathcal{P}_{LH}$.


## 1. Introduction

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}$, and let $B$ be the set of all functions $w(z) \in H(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w(z)\left(\frac{\overline{f}}{f}\right)f_z, \tag{1.1}$$

where $w(z)$ is the second dilatation of $f$ and $w(z) \in B$. It has been shown [1] that if $f$ is a non-vanishing log-harmonic mapping, then $f$ can be expressed as

$$f = h(z)\overline{g(z)} \tag{1.2}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$, i.e., $h(z), g(z) \in H(\mathbb{D})$. On the other hand, if $f$ vanishes at $z = 0$ but is not identically zero, then $f$ admits the following representation:

$$f = z|z|^{2\beta}h(z)\overline{g(z)} \tag{1.3}$$

where $Re\beta > -1/2$, $H(z), G(z) \in H(\mathbb{D})$ and $h(0) \neq 0$, $g(0) = 1$. In general, the class of log-harmonic mappings is denoted by $\mathcal{S}_{LH}$. Univalent log-harmonic mappings and log-harmonic mappings have been studied extensively (for details see [1–5]).

Let $\mathcal{P}_{LH}$ be the set of all log-harmonic mappings $R$ defined on the unit disc $\mathbb{D}$ which are of the form

$$R = H(z)\overline{G(z)}, \tag{1.4}$$

where $H(z)$ and $G(z)$ are in $H(\mathbb{D})$, $H(0) = G(0) = 1$ and $Re(R) > 0$ for all $z \in \mathbb{D}$. In particular, the set $\mathcal{P}$ of all analytic functions $p(z)$ in $\mathbb{D}$ with $p(0) = 1$ and $Rep(z) > 0$ in $\mathbb{D}$ is a subset of $\mathcal{P}_{LH}$ [2].

* Corresponding author.
*E-mail addresses:* e.ozkan@iku.edu.tr (H.E. Özkan), y.polatoglu@iku.edu.tr (Y. Polatoğlu).

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Finally, let $\Omega$ be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$, and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $F_1(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $F_2(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $F_1(z)$ is subordinate to $F_2(z)$ if there exists $\phi(z) \in \Omega$ such that $F_1(z) = F_2(\phi(z))$. We denote it by $F_1(z) \prec F_2(z)$ [6]. The following theorem was proved by Abdulhadi [2], and plays an important role in our study.

**Theorem 1.1.** *Let $R(z) = H(z)\overline{G(z)} \in \mathcal{P}_{LH}$. Then $p(z) = \frac{H(z)}{G(z)} \in \mathcal{P}$. Conversely, given $p(z) \in \mathcal{P}$ and $w(z) \in B$, there exist non-vanishing functions $H(z)$ and $G(z)$ in $H(\mathbb{D})$ such that $p(z) = \frac{H(z)}{G(z)}$, $R = H(z)\overline{G(z)} \in \mathcal{P}_{LH}$, and $R$ is a solution of (1.1) with respect to the given $w(z)$.*

In this paper, we will investigate the class of log-harmonic mappings defined by

$$\mathcal{P}_{LH(M)} = \left\{ R \mid R = H(z)\overline{G(z)} \in \mathcal{P}_{LH}, \ \left| \frac{H(z)}{G(z)} - M \right| < M, M \geq 1 \right\}.$$

## 2. The main results

**Theorem 2.1.** *$R(z) = H(z)\overline{G(z)} \in \mathcal{P}_{LH(M)}$ if and only if $\frac{H(z)}{G(z)} \prec \frac{1+z}{1-\left(1-\frac{1}{M}\right)z}$.*

**Proof.** Let $R(z) = H(z)\overline{G(z)}$ be an element of $\mathcal{P}_{LH(M)}$; then we have

$$\left| \frac{H(z)}{G(z)} - M \right| < M \Leftrightarrow \left| \frac{1}{M} \frac{H(z)}{G(z)} - 1 \right| < 1.$$

Therefore the function

$$\psi(z) = \frac{1}{M} \frac{H(z)}{G(z)} - 1$$

has modulus at most 1 in the unit disc $\mathbb{D}$ and so

$$\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\psi(z)} = \frac{\left(\frac{1}{M}\frac{H(z)}{G(z)} - 1\right) - \left(\frac{1}{M} - 1\right)}{1 - \left(\frac{1}{M} - 1\right)\left(\frac{1}{M}\frac{H(z)}{G(z)} - 1\right)}, \tag{2.1}$$

and then $\phi(0) = 0$, $|\phi(z)| < 1$; therefore by the Schwarz lemma,

$$|\phi(z)| \leq |z|. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$\frac{H(z)}{G(z)} = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{M}\right)\phi(z)}. \tag{2.3}$$

The equality (2.3) shows that

$$\frac{H(z)}{G(z)} \prec \frac{1 + z}{1 - \left(1 - \frac{1}{M}\right) z}.$$

Conversely, suppose that the functions $H(z)$ and $G(z)$ are analytic in $\mathbb{D}$, and satisfy the conditions $H(0) = G(0) = 1$, $\frac{H(z)}{G(z)} \prec \frac{1+z}{1-\left(1-\frac{1}{M}\right)z}$; then we have

$$\frac{H(z)}{G(z)} \prec \frac{1 + z}{1 - \left(1 - \frac{1}{M}\right) z} \Rightarrow \frac{H(z)}{G(z)} = \frac{1 + \phi(z)}{1 - \left(1 - \frac{1}{M}\right)\phi(z)} \Rightarrow$$

$$\frac{H(z)}{G(z)} - M = M \frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}.$$

On the other hand, the function $\left(\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)}\right)$ maps the unit circle onto itself; then we have

$$\left| \frac{H(z)}{G(z)} - M \right| = \left| M \frac{\frac{1-M}{M} + \phi(z)}{1 - \frac{1-M}{M}\phi(z)} \right| \leq M,$$