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Blow-up criterion for 3D viscous liquid–gas two-phase flow model

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ABSTRACT

This work deals with the blow-up criterion for the strong solution to the three-dimensional viscous liquid–gas two-phase flow model in terms of the $L^1(0,T;L^\infty)$-norm of the gradient of the velocity with two types of boundary conditions: the Dirichlet boundary condition and the Navier-slip boundary condition. There is no extra restriction on viscosity coefficients. The result also applies to the whole space case.

1. Introduction

In this paper, we consider the 3D viscous liquid–gas two-phase flow model in the following form

$$\begin{cases} m_t + \operatorname{div}(mu) = 0, \\ n_t + \operatorname{div}(nu) = 0, \\ (mu)_t + \operatorname{div}(mu \otimes u) + \nabla P(m,n) = \mu \Delta u + (\lambda+\mu)\nabla \operatorname{div} u \quad \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

with the initial conditions

$$(m,n,u)|_{t=0} = (m_0, n_0, u_0), \quad \text{in } \Omega, \tag{1.2}$$

and boundary conditions:

(i) Dirichlet boundary condition: $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, and

$$u = 0, \quad \text{on } \partial\Omega; \tag{1.3}$$

(ii) Navier-slip boundary condition: $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, and

$$u \cdot \tilde{n} = 0, \qquad \operatorname{curl} u \times \tilde{n} = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\tilde{n} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ is the unit outward normal to $\partial\Omega$.

The variables $m = \alpha_l \rho_l$, $n = \alpha_g \rho_g$, $u = (u^1, u^2, u^3)$ and $P = P(m,n)$ denote the liquid mass, gas mass, the velocity of the liquid and gas and the common pressure for both phases, respectively; μ and λ are viscosity constants, satisfying

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0. \tag{1.5}$$

The other unknown variables α_l and $\alpha_g \in [0,1]$ denote the liquid and gas volume fractions; ρ_l and ρ_g denote liquid and gas densities, satisfying equations of state

$$\rho_l = \rho_{l,0} + \frac{P - P_{l,0}}{a_l^2}, \qquad \rho_g = \frac{P}{a_g^2}, \tag{1.6}$$

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where a_l and a_g are known constants which denote, respectively, sonic speeds in the liquid and gas; $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants. Moreover,

$$\alpha_l + \alpha_g = 1. \tag{1.7}$$

Note that from (1.6) and (1.7), the pressure law satisfies

$$P(m, n) = C^0 \left(-b(m, n) + \sqrt{b(m, n)^2 + c(m, n)}\right), \tag{1.8}$$

here $C^0 = \frac{1}{2}a_l^2$, $k_0 = \rho_{l,0} - \frac{P_{l,0}}{a_l^2} > 0$, $a_0 = \left(\frac{a_g}{a_l}\right)^2$ and

$$b(m, n) = k_0 - m - \left(\frac{a_g}{a_l}\right)^2 n = k_0 - m - a_0 n,$$

$$c(m, n) = 4k_0 \left(\frac{a_g}{a_l}\right)^2 n = 4k_0 a_0 n.$$

We are interested in the Drift-flux type model of two-phase flows. The 1D version of the model, often combined with a more general slip law such that non-equal fluid velocities are taken into account, represents a useful model within petroleum and nuclear industry applications. In the present paper, we consider the simplified model of Drift-flux type (1.1), where we assumed that the two fluids have the common pressure and shared the equal velocity, neglected the external force and the effect of gas in the convective term in the mixture momentum equation. Such a (multi-dimensional) model is relevant to explore for various applications where the fluid is composed of gas that is dispersed in the liquid phase such that the two phases move with the same velocity more or less. For more information about the model, we refer the reader to [1–3] and references therein.

There are some works about the viscous liquid–gas two-phase flow model. For the model (1.1) in 1D, where the liquid is incompressible and the gas is polytropic, the global existence and uniqueness of weak solution to the free boundary value problem was studied in [4–7]. For more results about the 1D case of the relevant model, refer to [8–11], where more interesting phenomenon are described. Specifically, in [12], where both of the two fluids are compressible, the global existence of a weak solution is studied. For the model (1.1) in 2D, Yao et al. [13] obtained the existence of a global weak solution when the initial energy is small, and this can be viewed to be a generalization of the results in [12] from 1D to 2D. They [14] established a blow-up criterion in terms of the upper bound of the liquid mass for the strong solution in a smooth bounded domain when there is no initial vacuum. For the model (1.1) in 3D, Guo et al. [15] obtained the existence of the global strong solution when the initial energy is small, and the initial vacuum is allowed. Hou and Wen [16] proved that the bound of the $L_t^1 L_x^\infty$ norm of the deformation tensor of the velocity gradient $\mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ controlled the possible breakdown of the strong solutions with vacuum, when $0 \le \underline{s}_0 m_0 \le n_0 \le \bar{s}_0 m_0$, where $\underline{s}_0$ and $\bar{s}_0$ are positive constants. Recently, Wen et al. [17] have obtained a blow-up criterion in terms of the upper bound of the liquid mass for the strong solution with vacuum, and there is relaxed restriction $\frac{25\mu}{3} > \lambda$ on viscosity coefficients.

The method used to get the blow-up criterion of the strong solution to the viscous liquid–gas two-phase flow model is similar to the single phase Navier–Stokes equation. So, let us introduce some works about this for a single phase Navier–Stokes equation. For the 2D compressible Navier–Stokes equations, Sun and Zhang [18] obtained a blow-up criterion in terms of the upper bound of density for the strong solution. For the 3D compressible Navier–Stokes equations, they [19] obtained a blow-up criterion in terms of the upper bound of density for the strong solution, under the restriction $\lambda < 7\mu$. In both papers, the initial vacuum is allowed and the domain included both the bounded smooth domain and $\mathbb{R}^N$, $N = 2, 3$. Huang et al. [20] obtained Serrin type blow-up criterion for the strong solution. They [21] established the following blow-up criterion when there is initial vacuum and $\lambda < 7\mu$: if $T^* < \infty$ is the maximal time of the existence of the classical solution, then

$$\lim_{T\to T^*} \int_0^T \|\nabla u(t)\|_{L^\infty(\Omega)} dt = \infty.$$

Recently, Huang et al. in their paper [22] have removed the restriction $\lambda < 7\mu$ for a 3D model with initial vacuum, and got the blow-up criterion of the strong solution:

$$\lim_{T\to T^*} \int_0^T \|\mathcal{D}(u)(t)\|_{L^\infty(\Omega)} dt = \infty,$$

where $\mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla u^t)$.

In the present paper, we obtain a blow-up criterion for the strong solution to the 3D viscous liquid–gas two-phase flow model in terms of $L^1(0, T; L^\infty)$-norm of the gradient of the velocity without the extra restriction $\lambda < 7\mu$ in two types of boundary conditions, when there is no initial vacuum, which improved the result in [14], in which the result held for the 3D case under the restriction $\lambda < 7\mu$, and the result in the present paper can be applied to the Navier-slip boundary condition and the whole space case. We should mention that the ideas introduced by the authors in [18–20,22,23] play crucial roles in our proof here.