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Equilibria of nonconvex valued maps under constraints ☆

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ABSTRACT

In the paper the notion of n-tangency for set-valued maps defined on a subset of a Banach space is considered. The existence of equilibria of upper semicontinuous map being n-tangent to a sleek retract with the nontrivial Euler characteristic is established.

0. Introduction

Let M be a *compact* subset of a Banach space E and $\varphi : M \multimap E$ be an *upper semicontinuous* set-valued map with *compact* values. In the paper we ask about the existence of *equilibria* of φ, i.e. $x_0 \in M$ such that $0 \in \varphi(x_0)$.

A classical result due to Browder and Fan [4,9] says that if M is convex, $\varphi : M \multimap E$ has convex values and is *inward* in the sense that

$$\varphi(x) \cap T_M(x) \neq \emptyset \quad \text{for each } x \in M, \tag{1}$$

where $T_M(x) = \mathrm{cl}(\bigcup_{h>0} h(M - x))$, then φ admits an equilibrium. Observe that $T_M(x)$ is a tangent cone to M at x and therefore the inwardness condition (1) can be interpreted as a tangency condition.

This result has been generalized many times, e.g. [6–8]. In [2], Ben-El-Mechaiekh and Kryszewski relaxed the convexity of M and obtained a similar result. Namely, if M is *$\mathcal{L}$-retract* with the nontrivial Euler characteristic ($\chi(M) \neq 0$), φ is as above but tangent with respect to the Clarke cone, i.e.

$$\varphi(x) \cap C_M(x) \neq \emptyset \quad \text{for each } x \in M, \tag{2}$$

where $C_M(x)$ stands for the Clarke cone tangent to M at x, then an equilibrium still exists.

A natural problem concerning the relaxation of convexity of values of φ arises. As shown in [2], if M is as above and φ has acyclic (e.g. contractible or cell-like) values and satisfies the strong tangency condition

$$\varphi(x) \subset C_M(x) \quad \text{for each } x \in M, \tag{3}$$

then there equilibria exist.

The following conjecture was posed in [2]:

(*C*) *If M is an $\mathcal{L}$-retract such that $\chi(M) \neq 0$, φ has acyclic values and condition* (2) *is satisfied, then there exists an equilibrium of φ.*

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We show that the very conjecture is false.

Example 0.1. Let $M=[0,1]\times[0,1]$, $E=\mathbb{R}^2$ and $\varphi : M \multimap E$ be a map defined by:

$$\varphi(x,y) := \begin{cases} \operatorname{conv}(\{(-1,0),(0,1)\}) \cup \operatorname{conv}(\{(0,1),(1,0)\}), & \text{if } (x,y)\in\{1\}\times[0,1],\\ \{(1,0)\}, & \text{if } (x,y)\in[0,1)\times[0,1]. \end{cases}$$

Since M is convex, then for any $x\in M$, $C_M(x)=T_M(x)$ and $\varphi(x)\cap C_M(x)$ is nonempty and convex. Then condition (2) is satisfied and φ is upper semicontinuous with contractible, hence acyclic, values. Moreover M is compact and convex, thus M is an $\mathcal{L}$-retract and $\chi(M)=1$ (see Section 1). However, it is clear that $0\notin\varphi(x)$ for each $x\in M$.

Hence, it appears that the pointwise tangency condition (2) together with the acyclicity (or even contractibility) of values of φ are too weak for the existence of equilibria. In order to obtain a positive answer it seems that one needs to study the local behavior of φ with respect to M in terms of homotopical triviality. We provide a class of the so-called n-tangent set-valued maps with not necessarily convex values (see Definition 2.1) and show that for that class the problem of existence of equilibria has a solution (see Theorem 2.3, Corollary 2.7).

1. Preliminaries

We consider set-valued maps $\varphi : X \multimap Y$, where X and Y are metric spaces, that assign to each $x\in X$, a nonempty subset $\varphi(x)$ of Y. By the *graph* of φ we mean the set $\operatorname{Gr}(\varphi) := \{(x,y)\in X\times Y \mid y\in\varphi(x)\}$. We say that a set-valued map φ is *lower semicontinuous* if for any open set $U\subset Y$, the preimage $\varphi^{-1}(U) := \{x\in X\colon\ \varphi(x)\cap U\neq\emptyset\}$ is open; φ is *upper semicontinuous* if for any open set $U\subset Y$, the small preimage $\varphi^{+1}(U) := \{x\in X\colon\ \varphi(x)\subset U\}$ is open; φ is *continuous* if it is upper and lower semicontinuous simultaneously. By a *selection* of φ we mean a continuous map $f : X\to Y$ such that $f(x)\in\varphi(x)$ for any $x\in X$.

If $A\subset B$, then $A\hookrightarrow B$ is *homotopy n-trivial* provided that for any $0\leqslant k\leqslant n$, every continuous map $f_0 : S^k\to A$ admits a continuous extension $f : D^{k+1}\to B$, i.e. $f(x)=f_0(x)$ for any $x\in S^k$, where S^k and D^{k+1} stand for a unit sphere and a closed ball in $\mathbb{R}^{k+1}$. A map φ has *acyclic* values if $\check{H}^q(\varphi(x))\approx\check{H}^q(pt)$ for any $q\in\mathbb{Z}$ and $x\in X$, where $\check{H}$ denotes the Čech cohomology functor and pt is a one point space. In particular, if for any $x\in X$, $\varphi(x)$ is convex, contractible, cell-like, then for any $n=0,1,2,\ldots$, $\varphi(x)\in UV^n$,[1] and hence φ has acyclic values [10,3].

It is well known that approximation methods are helpful in the study of fixed points or equilibria of set-valued maps. However in the context of our problem we would like to look for a graph approximation $f : M\to E$ of φ satisfying the additional tangency condition: $f(x)\in C_M(x)$ for any $x\in M$. In [11, Th. 2.1] we have obtained a useful result in this direction. Below we recall an appropriate version of this result convenient for our purposes (comp. [11, Cor. 2.2, Rem. 2.3], [5,12]).

Theorem 1.1. *Let $n\geqslant 0$, X be a metric space, E be a Banach space, $\varphi : X\multimap E$ be upper semicontinuous with compact values, $C : X\multimap E$ be lower semicontinuous with closed and convex values. Then for any open neighborhood $\mathcal{U}$ of $\operatorname{Gr}(\varphi)$, there is a continuous selection $f : X\to E$ of C such that $\operatorname{Gr}(f)\subset\mathcal{U}$ provided that $\dim(X)\leqslant n+1$*[2] *and the following conditions hold*:

- (T) *for any $x\in X$, $\varphi(x)\cap C(x)\neq\emptyset$,*
- (C_n) *for any $x\in X$, for any open neighborhood U of $\varphi(x)$, there are an open neighborhood $V\subset U$ of $\varphi(x)$ and an open neighborhood W of x such that for any $y\in W$ the inclusion $V\cap C(y)\hookrightarrow U\cap C(y)$ is homotopy n-trivial.*

If condition (T) holds, then (C_n) is satisfied provided that φ has convex values. Moreover, if the strong tangency condition is satisfied, i.e. for any $x\in X$, $\varphi(x)\subset C(x)$, then (C_n) is equivalent to the condition: $\varphi(x)\in UV^n$ for any $x\in X$ (see [11, Lem. 2.13]).

In what follows we recall notions of tangent cones in a Banach space. Given a closed subset M of a Banach space E, for any $x\in M$, by

$$C_M(x) := \left\{ v \;\middle|\; \limsup_{t\to 0^+,\, x'\to_M x} \frac{d(x'+tv, M)}{t} = 0 \right\},$$

we denote the *Clarke tangent cone* to M at $x\in M$. It is well known that $C_M(x)$ is closed and convex and if M is convex, then $C_M(x)=T_M(x)$ (see [1]).

By $T_M^B(x)$ we denote the *Bouligand tangent cone* to M at x, i.e.

$$T_M^B(x) := \left\{ v \;\middle|\; \liminf_{t\to 0^+} \frac{d(x+tv, M)}{t} = 0 \right\}.$$

[1] Recall that for a subset A of metric space X, $A\in UV^n$ if for any open neighborhood U of A there is an open neighborhood $V\subset U$ of A such that the inclusion $V\hookrightarrow U$ is homotopy n-trivial.

[2] $\dim(X)$ denotes the covering dimension of the metric space X.