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On the second solution to a critical growth Robin problem

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ABSTRACT

We investigate the existence of the second mountain-pass solution to a Robin problem, where the equation is at critical growth and depends on a positive parameter λ. More precisely, we determine existence and nonexistence regions for this type of solutions, depending both on λ and on the parameter in the boundary conditions.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ ($n \geqslant 3$) be a smooth and bounded domain and let $2^* = \frac{2n}{n-2}$ be the critical Sobolev exponent. We consider the Robin problem

$$\begin{cases} -\Delta u = \lambda(1+u)^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u_\nu + cu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $c, \lambda > 0$ and u_ν denotes the outer normal derivative of u on $\partial\Omega$.

As pointed out in the seminal paper [9], the interest in problems like (1) is due to their similarity to some geometrical and physical variational problems where a lack of compactness also occurs (recall that the embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$ is not compact).

A solution u_λ to (1) is called *minimal* if $u_\lambda \leqslant u$ a.e. in Ω, for any other solution u to (1). Furthermore, we say that a solution u is *regular* if $u \in L^\infty(\Omega)$. From [5] we know

Proposition 1. *For every $c > 0$, there exists $\lambda^* = \lambda^*(c) > 0$ such that:*

(i) *for $0 < \lambda < \lambda^*$ problem (1) admits a minimal regular solution u_λ;*
(ii) *for $\lambda = \lambda^*$ problem (1) admits a unique regular solution u^*;*
(iii) *for $\lambda > \lambda^*$ problem (1) admits no solution.*

Furthermore, the map $c \mapsto \lambda^*(c)$ is strictly increasing and $\lambda^*(c) \to 0$, as $c \to 0$.

When $c = 0$, (1) reduces to the Neumann problem (for which no positive solutions exist), whereas the limit case $c \to +\infty$ may be seen as the Dirichlet problem. Indeed, Proposition 1 includes well-known results for the Dirichlet problem, see [9,13,16,19].

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Under Dirichlet boundary conditions, due to [9], we know that the equation in (1) admits, besides the minimal solution u_λ, a larger *mountain-pass* solution U_λ (see Section 2 for the definition) for every $\lambda \in (0, \lambda^*_{Dir})$, where λ^*_{Dir} is the extremal parameter for the Dirichlet problem. One of the purposes of the present paper is to investigate, for any $c > 0$ and $\lambda \in (0, \lambda^*(c))$, the existence of a larger mountain-pass solution U_λ to problem (1). This represents a further step towards a complete description of the set of solutions to (1).

Let $H(x)$ be the mean curvature of $\partial\Omega$ at x and let

$$H_{\max} := \max_{x \in \partial\Omega} H(x). \tag{2}$$

We show

Theorem 1. *Let $\lambda^*(c)$ be as in Proposition 1. For every $c > 0$, there exists $0 \leqslant \Lambda(c) < \lambda^*(c)$ such that problem (1) admits, besides the minimal solution u_λ, a mountain-pass solution U_λ for any $\Lambda(c) < \lambda < \lambda^*(c)$. Furthermore, the map $(0, +\infty) \ni c \mapsto \Lambda(c)$ is nondecreasing and the following statements hold:*

(i) *If $n = 3$ and $c > 0$ or $n \geqslant 4$ and $0 < c < \frac{n-2}{2} H_{\max}$, then $\Lambda(c) = 0$. Moreover, if $n = 4, 5$, then $\Lambda(\frac{n-2}{2} H_{\max}) = 0$.*
(ii) *If $n \geqslant 4$, there exists $K = K(\Omega) \geqslant \frac{n-2}{2} H_{\max}$ such that if $c > K$, then $\Lambda(c) > 0$, U_λ exists up to $\lambda = \Lambda(c)$ and does not exist if $0 < \lambda < \Lambda(c)$.*

Note that, arguing as in [6], any mountain-pass solution to (1) is regular. Hence, by elliptic regularity, it solves (1) in a classical sense.

When $\Lambda(c) > 0$, one may wonder if different kinds of solutions exist for $\lambda \in (0, \Lambda(c))$. If $\Omega = B$, the unit ball, in [5] explicit radial solutions to (1) have been determined for every $\lambda \in (0, \lambda^*(c))$. We briefly recall their construction. For $c > 0$ and $\eta > \eta_0(c)$, where

$$\eta_0(c) := \max\left\{0, \frac{n-2}{c} - 1\right\} \tag{3}$$

consider the function

$$\varphi(\eta) := \frac{[n(n-2)]^{n-2}}{c^4} \frac{[c(1+\eta) - n + 2]^4 \eta^{n-2}}{(1+\eta)^{2n}}. \tag{4}$$

It is readily seen that $\varphi(\eta_0) = 0 = \lim_{\eta \to +\infty} \varphi(\eta)$, that φ attains a global maximum at

$$\overline{\eta} := \frac{n + 2 + \sqrt{(n+2)^2 - 4c(n-2-c)}}{2c},$$

that φ increases on $(\eta_0, \overline{\eta})$ and decreases on $(\overline{\eta}, +\infty)$. Hence, for any $\lambda \in (0, \lambda_n(c))$, where $\lambda_n(c) := (\varphi(\overline{\eta}))^{1/(n-2)}$,

$$\text{there exist } \eta_i = \eta_i(\lambda, c) \quad (i = 1, 2) \quad \text{such that} \quad \varphi(\eta_i) = \lambda^{n-2}. \tag{5}$$

If $\lambda = \lambda_n(c)$, then $\eta_1 = \eta_2 = \overline{\eta}$. Finally, we recall by [5]

Proposition 2. *Let $\Omega = B \subset \mathbb{R}^n$ $(n \geqslant 3)$. Then, if $\lambda_n(c) > 0$ and $\eta_0 < \eta_2 \leqslant \overline{\eta} \leqslant \eta_1$ are defined as in (5), we have*

(i) *for every $\lambda \in (0, \lambda_n(c))$, there exist two radial solutions of problem (1), the minimal solution u_{η_1} and a larger solution u_{η_2}, given by*

$$u_{\eta_i}(x) = \left(\frac{n(n-2)\eta_i}{\lambda}\right)^{(n-2)/4} \left(\eta_i + |x|^2\right)^{-(n-2)/2} - 1, \quad i = 1, 2;$$

(ii) *the extremal parameter satisfies $\lambda^*(c) = \lambda_n(c)$ and the extremal solution u^* of (1) is given by $u^*(x) := u_{\overline{\eta}}(x)$.*

Letting $c \to +\infty$ in Proposition 2, one recovers known results for the corresponding Dirichlet problem, see [16, Section 5]. In particular, $\lambda_n(c) \nearrow \lambda^*_{Dir}$, see also [19, Section VI].

In Section 4 we show that the larger solution u_{η_2} in Proposition 2 has high energy when $c > \frac{n-2}{2}$ and λ is sufficiently small. Combining this with the fact that u_{η_1} and u_{η_2} are the only radial solutions to (1), we prove

Theorem 2. *Let $\Omega = B \subset \mathbb{R}^n$ $(n \geqslant 3)$ and $\lambda_n(c)$ be as in Proposition 2. Then*

(i) *if $0 < c \leqslant \frac{n-2}{2}$, problem (1) admits, besides the minimal solution, a radial mountain-pass solution U_λ for every $0 < \lambda < \lambda_n(c)$;*