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Uniqueness and value distribution of differences of entire functions ☆

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ABSTRACT

We consider the existence of transcendental entire solutions of certain type of non-linear difference equations. As an application, we investigate the value distribution of difference polynomials of entire functions. In particular, we are interested in the existence of zeros of $f^n(z)(\lambda f^m(z+c)+\mu f^m(z))-a$, where f is an entire function, n, m are two integers such that $n \geqslant m > 0$, and λ, μ are non-zero complex numbers. We also obtain a uniqueness result in the case where shifts of two entire functions share a small function.

1. Introduction

A meromorphic function means meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). We assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [7,18]. We use $\sigma(f)$ to denote the order of f and $N_p(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m \leqslant p$ and p times if $m > p$. For a small function a related to f, we define

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)}.$$

Recently, Yang and Laine [19] considered the existence of the solutions of a non-linear differential-difference equation of the form

$$f^n + L(z, f) = h, \tag{1}$$

where $L(z, f)$ is a linear differential-difference polynomial in f. They obtained the following result.

Theorem A. *(See [19, Theorem 3.4].) Let P, Q be polynomials. Then a non-linear difference equation*

$$f(z)^2 + P(z)f(z+1) = Q(z)$$

has no transcendental entire solution of finite order.

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Theorem B. *(See [19, Theorem 3.5].) A non-linear difference equation*

$$f(z)^3 + P(z)f(z+1) = c\sin bz \tag{2}$$

where $P(z)$ is a non-constant polynomial and $b, c \in \mathbb{C}$ are non-zero constants, does not admit entire solutions of finite order. If $P(z) = p$ is a non-zero constants, then (2) possesses three distinct entire solution of finite order, provided $b = 3n\pi$ and $p^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a non-zero integer n.

Laine and Yang [9] continued to consider the non-existence of transcendental entire solutions of non-linear differential equations of type

$$f^n + P_d(f) = p_1 e^{a_1 z} + p_2 e^{a_2 z}, \tag{3}$$

and obtained new results which complementing the theorems given by Li and Yang [10,11].

Theorem C. *(See [9, Theorem 3.1].) Let $n \geqslant 3$ be an integer and $P_d(f)$ be a differential polynomial in f of total degree $d \leqslant n-2$ with polynomial coefficients such that $P_d(0) = 0$. Provided that p_1, p_2 are non-vanishing polynomials and a_1, a_2 are distinct non-zero complex constants, then (3) has no transcendental entire solutions.*

Laine and Yang [9] pointed out that a similar conclusion could be proved if the differential polynomial $P_d(f)$ is replaced with a differential-difference polynomial. However, Theorems A, B and C, the degree of the differential-difference polynomial is less than n. Now, we consider the equal-case, we get the following results.

Theorem 1. *Let a, c be non-zero constants, n and m be integers satisfying $n \geqslant m > 0$, $\lambda \neq 0$ be a complex number and let $P(z)$, $Q(z)$ be polynomials. If $n \geqslant 2$, then the difference equation*

$$f(z)^{n+m} + \lambda f(z)^n f(z+c)^m = P(z)e^{Q(z)} + a \tag{4}$$

has no transcendental entire solutions of finite order.

Remark. It seems to us that replacing $f(z)^n f(z+c)^m$ with $f(z)^n \sum_{j=1}^m f(z+c_j)$ and $c_j \neq 0$, or replacing the non-zero value a with $a(z) \not\equiv 0$, where $a(z)$ is a polynomial in z, the same conclusion can be proved.

Let f be a transcendental meromorphic function, and let n be a positive integer. Reminiscent to the value distributions of $f^n f'$, Hayman [5, Corollary to Theorem 9] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geqslant 3$. Mues [15, Satz 3] proved that $f^2 f' - 1$ has infinitely many zeros. Later on, Bergweiler and Eremenko [1, Theorem 2] showed that $ff' - 1$ has infinitely many zeros also. Corresponding to the results above, Laine and Yang [8, Theorem 2] investigated the value distribution of difference products of entire functions.

Theorem D. *(See [8, Theorem 2].) Let f be a transcendental entire function with finite order, and let c be a non-zero complex constant. Then, for $n \geqslant 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often.*

Some improvements of Theorem D can be found in [12,13]. In the present paper, we consider the value distribution of $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$, where n, m are non-negative integers, and λ, μ are non-zero complex numbers. We obtain the following result which generalize some theorems in [8,12,13].

Theorem 2. *Let f be a transcendental entire function with finite order, c be a non-zero constant, n and m be integers satisfying $n \geqslant m > 0$, and let λ, μ be two complex numbers such that $|\lambda| + |\mu| \neq 0$. If $n \geqslant 2$, then either $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often or $f(z) = e^{\frac{\log t}{c}z}g(z)$, where $t = (-\frac{\mu}{\lambda})^{\frac{1}{m}}$, and $g(z)$ is periodic function with period c.*

Remarks. (1) If $m = 0$ and $\lambda + \mu \neq 0$, then $(\lambda + \mu)f^n$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often provided that $n \geqslant 2$.

(2) It seems to us that replacing the non-zero value $a \in \mathbf{C}$ with $a(z) \not\equiv 0$, where $a(z)$ is a polynomial in z, a similar conclusion can be proved.

(3) When $m > n > 0$. If $\lambda\mu = 0$ and $|\lambda| + |\mu| \neq 0$, $m \geqslant 2$, then Theorem 2 holds. Unfortunately, when $\lambda\mu \neq 0$, $m \geqslant 2$, we do not know whether Theorem 2 holds.

(4) When $\lambda\mu \neq 0$, $m = 1$ and $n = 0$, we can give a counterexample. Namely, let $f(z) = z + e^z$, $\lambda = 1$ and $\mu = -1$. Then $f(z+c) - f(z) = c$, where $c = 2\pi i$. Clearly, $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$ cannot assume every non-zero value $a \in \mathbf{C}$ infinitely often.